

ACIC TECHNICAL REPORT NO. 86

AD 622335

OCTOBER 1959



BLANK PAGES DELETED

Determination of Ground Positions From Observations of Artificial Earth Satellites

AERONAUTICAL CHART AND INFORMATION CENTER
AIR PHOTOGRAPHIC AND CHARTING SERVICE (MATS)
UNITED STATES AIR FORCE,
ST. LOUIS 18, MO.

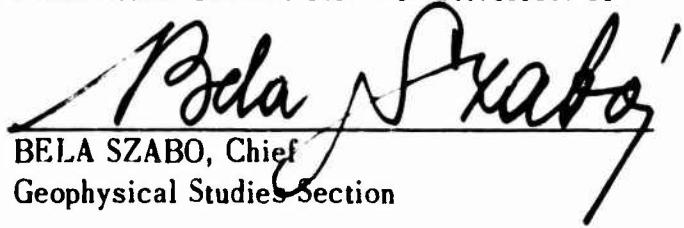
DETERMINATION OF GROUND POSITIONS FROM OBSERVATIONS
OF ARTIFICIAL EARTH SATELLITES

ACIC Technical Report No. 86

OCTOBER 1959

Prepared by
Geo-Sciences Branch
Geophysical Studies Section

PREPARED UNDER THE SUPERVISION OF:


BELA SZABO, Chief
Geophysical Studies Section

APPROVED:


ROBERT E. HERNDON, Jr., Colonel USAF
Commander

AERONAUTICAL CHART AND INFORMATION CENTER
Second and Arsenal Streets
St. Louis 18, Missouri

NOTICES

This report is issued to disseminate timely, useful technical procedure to activities engaged in geodesy and related subjects. Nothing in this report is to be construed as necessarily coinciding with USAF doctrine.

This report was prepared under the direction of the Aeronautical Chart and Information Center, USAF. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

ABSTRACT

The purpose of this study was to devise workable procedure for determining the geodetic position of an observation site from orbital data and photographic plates of artificial satellites.

Development of procedure depended heavily on published information from the Smithsonian Institute, Space Track, and the U. S. Naval Observatory.

Along with Space Track bulletins, two or more good photographic plates of a satellite are needed for a solution.

The time lapse between plates should be at a minimum. This recommendation stems from a conclusion that satellites now in orbit are not entirely satisfactory for geodetic purposes because their orbital elements are changing too much from day to day.

TABLE OF CONTENTS

	Page No.
INTRODUCTION	1
THE DETERMINATION OF RIGHT ASCENSION AND DECLINATION OF AN ARTIFICIAL EARTH SATELLITE AT A GIVEN UNIVERSAL TIME FROM ORBITAL DATA.....	3
Time.....	3
Elements of the Celestial Sphere.....	5
Development of Formulae with Perturbations Not Considered.....	7
Development of Formulae with Perturbations Considered	7
Determination of the Period	9
Procedure.....	9
Example.....	11
THE DETERMINATION OF RIGHT ASCENSION AND DECLINATION OF AN ARTIFICIAL EARTH SATELLITE FROM PHOTOGRAPHIC OBSERVATION..	14
Identification and Selection of Reference Stars.....	14
Determination of the Plate Center.....	17
Plate Measurements.....	20
Computation of the Standard Coordinates of the Reference Stars	20
Computation of the Standard Coordinates of the Unknown Point.....	21
Celestial Coordinates of the Satellite.....	26
Procedure.....	26
Example.....	28
THE DETERMINATION OF THE GEOCENTRIC COORDINATES OF A STATION FROM SATELLITE OBSERVATIONS	32
Methods of Finding the Observer's Coordinates	34
Procedure.....	36
Examples.....	37
APPENDIX A - The Ellipse and Spherical Right Triangle	44
APPENDIX B - Basic Elements of a Satellite's Orbit	46
APPENDIX C - The Earth's Oblateness from Satellite Observation.....	50
APPENDIX D - Derivations of Formulae for Photographic Plate Reductions.....	51
APPENDIX E - Discussion of Errors Pertaining to the Photographic Method.....	58

TABLE OF CONTENTS (Continued)

	Page No.
APPENDIX F - Development of the Method of Dependences.....	59
APPENDIX G - Development of Formulae for the Geocentric Coordinates of the Observation Station.....	61
APPENDIX H - Relationship between Geocentric, Geodetic, and x, y, z Space Coordinates.....	67
REFERENCES	70

INTRODUCTION

The uses for artificial earth satellites are increasing and expanding into more scientific fields than was first expected. The differing sizes and shapes of the orbits as well as the varying instrumentation provide data for a wide range of scientific fields.

Observations of a satellite can help the geodesist to determine some very basic and essential data. A study of the irregularity of the orbit can reveal data concerning the earth's gravitational field. From investigations of a satellite's period and acceleration, more precise data can be collected from which to determine the size of the earth. The rate of change of the plane of the orbit will help fix the flattening of the earth. Finally, the satellite provides the geodesist with a direct method for connection of datums.

The field of geodesy can use the satellites in two basic ways. One method involves observing the satellite simultaneously from two or more stations. The timing must be precise. Atmospheric refraction ceases to be important due to the simultaneity of exposure.

The other method involves knowing the orbit of the satellite and then making observations from any station on the earth's surface. The latter method will be the one treated in this report. The requirement of the satellite for the carrying out of this method is largely in the character of the orbit rather than in special instrumentation. A steady circular orbit would provide the necessary conditions for a satellite to be used in triangulation for establishing the geodetic position of an observation site or for the connection of datums.

Most geodetic needs depend on accurate prediction of the satellite's position; something which is not possible with an orbit of large eccentricity. The perturbing factors such as the earth's gravity and atmospheric drag are more pronounced on a satellite the closer it is to the earth. On the other hand, a satellite which is a great distance from the earth may be very difficult to track either photographically or electronically.

It is highly unlikely that a circular orbit could be obtained. There is no example of any orbiting body traveling in a circular path in the known universe. But we can expect to attain an elliptical orbit with a very small amount of flattening which would be approaching the character of a circle.

The steadiness of the orbit is a necessity for geodetic purposes because in order to make use of the satellite as a celestial triangulation point, its position must be calculated from known orbital data. If the orbit is changing greatly from one revolution to the next, the data applicable for computing the position during one revolution will not be true for computations of any position in the next revolution.

When the character of the orbit more nearly satisfies the geodetic requirements, then the remaining consideration in connection with photographic tracking is visibility. The camera has to be able to record the image or the path of the satellite on the photographic plate. The size and height of the satellite as well as characteristics of the orbit will determine the visibility photographically.

Especially designed equipment is being made for photographic satellite tracking and

methods of plate reduction specifically adapted to each type of equipment have been or are being worked out. The Markowitz Dual Rate Tracking camera, the Schmidt camera, and the Baker-Nunn Tracking camera are some of the instruments which have been designed especially for gathering satellite data.

As is true of most scientific equipment, the portability decreases with the precision of the machinery. Most of the specially designed equipment cannot be considered portable field equipment in the sense that it can be transported and assembled in a matter of days. However, the Markowitz camera is considered feasible for observation stations maintained for six months to a year.

It is the purpose of this report to present a method for finding the coordinates of a position on the earth from data obtained from observations of an artificial earth satellite. Included in the intention is the recognition of the need for a method which is usable with portable equipment.

The presentation is organized in three phases. The first phase presents the formulae and procedure for computing the coordinates of the satellite from published orbital data. Agencies with the assignment of computing the orbits of satellites collect information from their various field stations and compute and publish the necessary orbital elements such as the inclination, right ascension of the ascending node, the distance at apogee and perigee, et cetera, from which the coordinates may be obtained. Smithsonian Astrophysical Observatory and Air Force Cambridge Research Center are the two agencies which supply the orbital data and prediction information to ACIC.

The second phase describes the adaptation of classical methods of astronomical photography for the purpose of obtaining the coordinates of the satellite from a photographic plate. It is not the intention of this report to describe the method of photographing the satellite or recording the time of observation. Phase two begins with the procedure for identifying the reference stars after the photographic plate has been processed. It must be assumed that the operator is trained in the use of the measuring instrument. The reduction is prepared for readings from a linear comparator type of plate measuring device; not for readings of angular measure.

The third phase combines the results of the first and second phases in order to produce the x, y, z space coordinates of the observation site. These can be converted to geocentric latitude and longitude. Then these geocentric coordinates can be converted into geodetic latitude and longitude by the standard formulae which has been included in Appendix F.

These three phases of the report contain all the necessary working formulae, the outlined procedure, and sample computations. The derivation of formulae and additional information which is not essential to the computation procedure is relegated to the appendices.

THE DETERMINATION OF RIGHT ASCENSION AND DECLINATION OF AN ARTIFICIAL EARTH SATELLITE AT A GIVEN UNIVERSAL TIME FROM ORBITAL DATA

The orbit of a satellite traveling around the earth is an ellipse with the earth as one of the foci. This elliptical path is undergoing continuous changes due to gravitational pull from the earth, atmospheric drag, and non-sphericity of the earth. The effects of these forces on the satellite's orbit are referred to as perturbations. The result of these combined perturbing factors can be determined by observing what has actually happened to the orbit of a satellite by comparing successive revolutions. But it is much harder to predict what their effect will be on future revolutions of the satellite. Other elements, too, make predictions more difficult: the proximity to the earth, the ellipticity of the orbit, and the speed of the satellite. The closer the satellite travels to the earth, the more its orbit is affected by the earth's gravitational attraction; also the lower it is in the earth's atmosphere the more it is perturbed by the atmospheric drag. A very eccentric orbit causes the satellite to travel close to the earth in part of the orbit and many times farther away from the earth in the opposite part of the orbit. The disturbing effects are not constant, then, throughout the path. More disturbances are influential at the close-in point, called perigee, than at the farthest-away point, the apogee. The greater the speed of any space vehicle, the less its path is diverted from stability. (The speed can vary from the minimum which just barely overcomes the force tending to draw a mass back to earth to a maximum value which allows the space object to completely overcome the earth's gravitational pull, and escape from an orbit around the earth.)

Many other forces with lesser effects also contribute to the difficulty of determining orbital elements which will hold true for more than just a few revolutions. It is necessary to realize that the orbital elements are undergoing continual change, when the position of the satellite at a particular time is to be computed from the orbital data. If a position of the satellite is desired for a particular time which relates to the 1500th revolution, for example, the coordinates of the position will be more dependable if the orbital elements refer to that same revolution than if the orbit was computed from the 1480th revolution. How much the orbit varies from revolution to revolution, of course depends on the characteristics of the particular satellite.

Care should therefore be taken to select observations of the satellite which have been made during (or as closely as possible to) a revolution for which orbital data was computed.

In order to compute the coordinates of the satellite from orbital data we have to consider the elements involved.

Time

When the sun is on the meridian of a place it is apparent noon there; when the sun is next on the meridian, an apparent solar day is said to have elapsed. The meridian of Greenwich is used as the reference point or the 0 meridian and the solar time of the Greenwich meridian is referred to as Greenwich Civil Time (GCT).

Starting from Greenwich the rest of the world is divided into 24 zones. In each zone, a standard time appropriate to a definite meridian within the zone is kept. This standard time is also referred to as zone time. This local zone time can be converted to GCT by adding or subtracting the value of the longitude of the central meridian in time. For example, the central meridian of Central Standard Time is 90°W . which is equivalent to $+6^{\text{h}}$. GCT is then the local zone time plus 6^{h} . (West longitude is added; East longitude is subtracted)

By agreement GCT is also called Universal Time (UT). In referring to the satellite, Universal Time is used. UT is counted from 0 to 24 hours, beginning at Greenwich midnight.

Observation date (or epoch) is usually expressed in decimal parts of a day. For example an observation date of August 21, 1958 at 10 hours 23 minutes 15.65 seconds in the morning would be written:

1958 August 21 $10^{\text{h}}23^{\text{m}}15.65^{\text{s}}$
or
1958 August 21.432820

Elements of the Celestial Sphere

The reference surface or celestial sphere is considered to be a sphere of infinite radius whose center coincides with the center of the earth. All astronomical bodies are thought of as being located on the celestial sphere, regardless of their distances from the earth's center. It is fixed in space, and since the earth rotates, the celestial sphere appears to rotate from east to west.

The north and south celestial poles represent the intersections of the earth's axis of rotation with the celestial sphere. The extension of the plane of the earth's equator intersects the celestial sphere in the great circle called the celestial equator.

A celestial meridian is an arc on the celestial sphere which is the extension of any terrestrial meridian.

The ecliptic is the great circle on the celestial sphere which is the apparent annual path of the sun.

The plane of the ecliptic is at an angle with the plane of the equator. This angle is called the obliquity of the ecliptic and is approximately $23\frac{1}{2}$ degrees. The two points where the ecliptic and equator intersect are called the equinoxes.

The vernal equinox (Γ) is the point at which the path of the sun crosses the equator going from the southern to the northern side.

To describe the position of any celestial object on the celestial sphere we use a system of coordinates which relate the position to the vernal equinox and the celestial equator. Right ascension (α) is the angular distance measured eastward from the vernal equinox along the celestial equator to the point where the celestial meridian through the

object intersects the equator. The declination (8) is the angular distance from the equator to the object measured along the meridian.

Declination is never greater than 90° , but it is positive north of the equator and negative south of the equator. Right ascension is positive and is generally given in terms of time. The relationship between time and arc is:

$$1^h = 15^\circ$$

$$1^m = 15'$$

$$1^s = 15''$$

The hour circle is a great circle intersecting the celestial equator at right angles and containing the north and south celestial poles and the celestial object. It can be considered the same as the celestial meridian. It is apparent that every celestial object will have an hour circle.

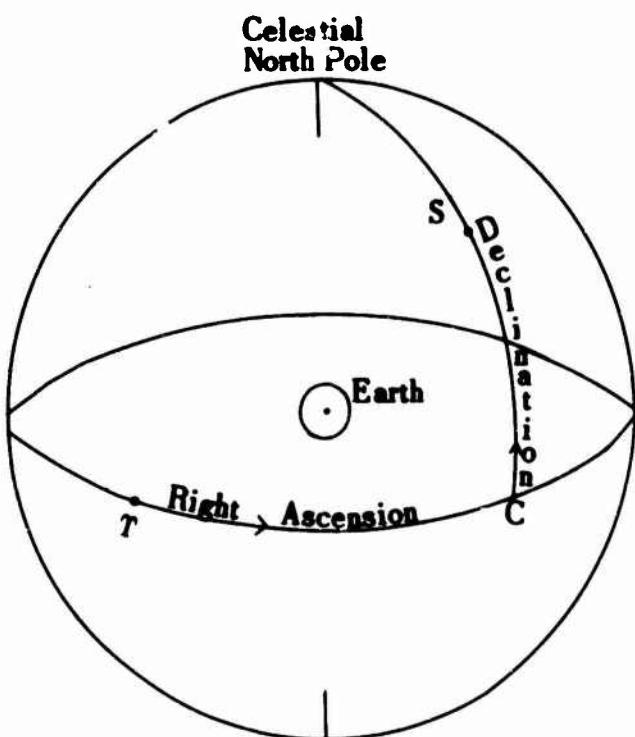


Fig. 1
Elements Pertaining to Elliptical Orbits

As previously stated, the path of a satellite around the earth is an ellipse with the earth as one of the foci. The apogee and perigee distances are respectively the maximum and minimum distances of a satellite from the earth.

Referring to Fig. 2: F represents the earth's center; A, the point of perigee; A', the point of apogee. The imaginary line A'A is the major axis of the orbit. One half of A'A is the semimajor axis designated by the letter a . If S is the position of the satellite at any point in the orbit, then the distance FS is the radius vector R . Since the orbit is elliptical, the value of R will not be constant; therefore, in order to designate the position in the orbit to which any particular R refers, we must specify the Universal

Time of the position. Two angles to consider at this time are the true anomaly (v) and the mean anomaly (M). Angle AFS is the true anomaly or the angular distance of the satellite from perigee measured along the orbit. If the angle AFS would be visualized as an angle that is zero at perigee and increases uniformly at a rate of 360 degrees per orbital period, it would then be correctly called the mean anomaly.

Referring to Fig. 3:

F is again the earth's center; S and A are projections of the satellite and perigee point on the celestial sphere; D is the intersection of the satellite's meridian (hour circle) with the celestial equator; N is the projection on the celestial equator of the ascending node where the satellite, going in a northward direction in its orbit, crosses the equator.

Two elements specify the orientation in space of the orbital plane; the inclination, i , and the right ascension of the ascending node designated with the Greek letter, capital Omega (Ω). The inclination is the angle between the orbital plane and the earth's equatorial plane. Ω is the arc (TN), measured eastward along the celestial equator, from the vernal equinox (T) to the ascending node (N).

Another orbital element, "the argument of the perigee" (ω), a small omega, is the angle, as seen from F, from N to the perigee point. The angle ω lies in the orbital plane, and is measured in the direction of the satellite's motion.

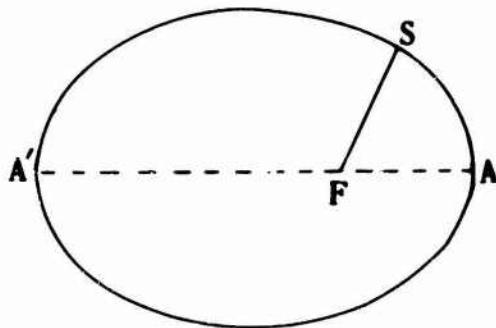


Fig. 2

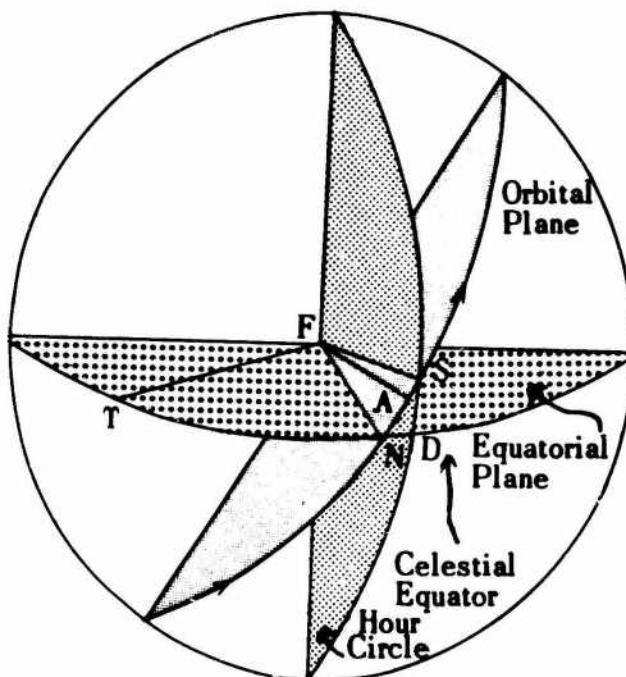


Fig. 3

Finally, three other elements, the orbital period (P), the "epoch of perigee" (T), and the eccentricity of orbit (e) must be mentioned. If the orbit is ideal and unchanging, P is simply the interval between successive passages through any selected fixed point of the orbit. In this phase, the selected point will be the ascending node and the period (nodical) will be determined for a specific time (UT). Because the orbits will not be ideal, the change in the nodical period with respect to a change in time will be roughly computed representing the average daily change. T is the time of passage through perigee. e along with a are the two elements which specify the size and shape of the orbit (see Appendix A).

The true anomaly (v), in fig. 3, would be angle AFS, provided you consider the angle has not been increasing uniformly. Arc SD represents the declination (δ) and the arc TD, the right ascension (α).

Development of Formulae with Perturbations Not Considered

Once the elements a , e , i , Ω_t , ω_t , T , and P are known, the n_t , R_t , v_t , δ , and α can be predicted for a particular time (t). n is the mean angular velocity of a satellite in its orbit expressed in radians per second. A value for n is given by:

$$n_t = \frac{2\pi}{P_t}, \text{ see Appendix B, part 3.}$$

An approximate formula for R_t would be:

$R_t = a [1 - e \cos n_t (t - T)]$, as given by Davis, Whipple, and Zirker¹. T must be the time of the perigee passage nearest to t , time wise, but preceding t . Then for v_t :

$$\cos v_t = \frac{a(1-e^2) - R_t}{e R_t}, \text{ the polar equation for an ellipse. (see Appendix A, part 1d).}$$

Next the declination and right ascension are given by:

$\sin \delta = \sin i \sin (\omega_t + v_t)$, where ω_t and v_t indicate values of ω and v for an observation time t .

$\alpha = \Omega_t + \text{arc tan} [\cos i \tan (\omega_t + v_t)]$, see Appendix A, part 2a and b for the basic spherical trigonometry formulae.

It should be emphasized that the two formulae just given apply only for an observer at the center of the earth, or to one directly beneath the satellite.

Development of Formulae with Perturbations Considered

It is well known that the Earth's oblateness and its atmosphere cause the main perturbations to the orbit of a near satellite, and various theoretical studies of these effects have been made. Fortunately, it happens that the first order perturbations due to the atmosphere and the Earth's oblateness are of different kinds; the air drag steadily reduces the length of the major axis and the eccentricity of the orbit, while the Earth's oblateness causes the orbital plane to rotate about the Earth's axis and the major axis of the orbit to

rotate in the orbital plane. The rotation of the major axis, expressed by changes in ω , is difficult to determine, but the rotation of the orbital plane, expressed by changes in Ω , can be measured very accurately.

To determine how much the right ascension of the ascending node (Ω) changes in radians each second of time, the following expression is taken from Merson and King-Hele:⁸

$$\text{Change in } \Omega = n B^2 \cos i [J + J^2 B^2 (\frac{19}{12} \sin^2 i - 1) - \frac{3}{14} D B^2 (7 \sin^2 i - 4)]$$

In the expression above:

$$B = \frac{r_e}{a(1-e^2)}$$

where:

r_e = Earth's equatorial radius

J, D = Constants which are functions of the Earth's oblateness and centrifugal and gravitational acceleration at the equator. If based on an oblateness of $\frac{1}{297}$:

$$J = 0.001637$$

$$D = 10.6 \times 10^{-6}, \text{ see Appendix C for further explanation of } J \text{ and } D.$$

Information published by the Astrophysical Observatory of the Smithsonian Institution in Cambridge, Massachusetts in the form of "Announcement Cards" and bulletins published by Space Track will list a simplified formula for determining the rate of change in Ω for a particular satellite. The example in this phase includes such a formula.

As has been previously stated, when computing the changes in ω (argument of perigee) it amounts to determining the extent of rotation of the major axis of the orbit in the orbital plane which is beyond the scope of this paper.

The rate of change in ω as found in Space Track bulletins incorporates the time of the passage through the ascending node (T_N). As an example of how this is presented, the following statement is taken from Space Track bulletin #158 for the satellite 1958 Delta II:

"Argument of Perigee 287.618 Degrees Minus .3668 Paren T Minus Epoch Paren."

To further explain this statement, it should be written in mathematical form, as follows:

$\omega_t = \omega_0 - 0.3668(T_N - t_0)$, the T in the statement is actually T_N and not the perigee passage, where

ω_t = the argument of perigee which is nearest, time wise, to an observation time t

ω_0 = 287.618, the value computed for the time t_0

T_N = the time of a passage through the ascending node N which is nearest, time wise, to t

t_0 = called "epoch" in the statement, refers to that observation time for which the elements $\omega_0, \Omega_0, a_0, e_0, P_0$, etc. were computed

The formula for R_t given by Moulton² is:

$R_t = a(1 - e \cos E_t)$, the development of this formula is given in Appendix B, part 4.

where

$$E_t = n_t (t - T) + e_t \sin n_t (t - T) + \frac{e_t^2}{2} \sin 2 n_t (t - T)$$

or

$$E_t = M_t + e_t \sin M_t + \frac{e_t^2}{2} \sin 2 M_t$$

The first term M_t must be expressed in radians. E is called the "eccentric" anomaly (see Appendix B, part 4).

The formula for v_t given by Sterne⁷ is:

$$\tan 1/2 v_t = \left[\frac{1 + e_t}{1 - e_t} \right]^{1/2} \tan 1/2 E_t$$

Determination of the Period

Space Track includes in its satellite bulletins a tabulated listing of the Greenwich meridian time (U.T.) for the south-north equatorial crossings. This listing may include the crossings over a period of four to five days. Since these times are for the south-north crossings of a satellite, they refer to the passage through the ascending node.

When the time for a passage through the ascending node is used in a formula, it is usually denoted by T_N (which is similar to the symbol T used for the time of a passage through perigee).

To determine the period P_t of a satellite for a particular observation time t , refer to the tabulated listing and locate the T_N nearest (time-wise) to t . Calling this the "center" T_N , find the difference between it and the preceding T_N and also the difference between it and the next T_N . The best value for P_t would then be the average of the two differences. The period found in this manner refers to the nodical period.

Procedure

This section will provide an outline and examples, based on Space Track data, to follow for determining the right ascension and declination of an artificial satellite, as seen from the center of the earth, for a given observation time t (U.T.).

1st - List the known orbital data such as a_o , e_o , i , ω_o and Ω_o which will be given for a particular time t_o .

2nd - List the formulae for determining a_t , e_t , ω_t and Ω_t for time t . Also list the nearest nodal passage (T_N) to time t and list P_t .

The formulae for a_t and e_t will be in the form $a_o \pm$ the daily change and $e_o \pm$ the daily change (see the example). The daily change in these elements is usually minus and may be determined by use of two or three successive bulletins which have these

elements listed along with the corresponding times (t_o).

The formulae for ω_t and Ω_t involve T_N and t_o and must be taken from the same bulletin which gave ω_o and Ω_o (ω_o and Ω_o being the first term of each formula).

3rd - Tabulate the following terms: $t - t_o$, $T_N - t_o$, $(T_N - t_o)^2$, a_t , e_t , e_t^2 , $\left(\frac{1+e_t}{1-e_t}\right)^{\frac{3}{2}}$, $\sin i$, $\cos i$, and $n_t \left(\frac{2\pi}{P_t}\right)$.

4th - Compute Ω_t , ω_t , and v_t . (for v_t use the formula given by Sterne)

The computation of v_t will involve computing values for T , M_t and E , in that order. T , once again, refers to the U.T. of the passage through perigee that is nearest (time-wise) to time t . T is not stated in Space Track bulletins, but a simplified method of computing T is a part of the examples which follow this procedure outline. (If Smithsonian bulletins are available, they will include a direct formula for finding T . The only unknown in this particular formula is the number of revolutions which the satellite has made between the time used in the formula and time t .)

5th - Compute R_t using the formula given by Moulton.

6th - Compute δ and α

$$\sin \delta = \sin i \sin (\omega_t + v_t)$$

$$\alpha = \Omega_t + \text{arc tan} [\cos i \tan (\omega_t + v_t)]$$

EXAMPLE

Given:

1st - $a_o = 1.128720$ earth radii (this means 1.128720 times an accepted value for the equatorial radius of the earth)

$$e_o = 0.085822$$

$$i = 65^\circ 12'$$

$$t_o = \text{Aug. } 24.912810 \text{ UT}$$

$$\omega_o = 28.876$$

$$\Omega_o = 105.771$$

$$P_o = 101.50 \text{ minutes}$$

2nd - $t = \text{Aug. } 25.0774535$

$$T_N = \text{Aug. } 25.0530313 \quad (\text{nearest nodal passage to } t - \text{time wise})$$

$$a_t = a_o - 0.000442 (t - t_o) \quad (\text{average daily change})$$

$$e_t = e_o - 0.000358 (t - t_o) \quad (\text{average daily change})$$

$$\omega_t = \omega_o - 0.350 (T_N - t_o) \quad (\text{from bulletin})$$

$$\Omega_t = \Omega_o - 2.782 (T_N - t_o) - 0.0018 (T_N - t_o)^2$$

$$P_t = 101.45 \text{ minutes}$$

3rd - Tabulation:

$$t - t_o = 0.1646435 \quad a_t = 1.128647 \text{ earth radii}$$

$$T_N - t_o = 0.1402213 \quad e_t = 0.085763$$

$$(T_N - t_o)^2 = 0.019662 \quad e_t^2 = 0.007355$$

$$\sin i = 0.907777 \quad \left[\frac{1+e_t}{1-e_t} \right]^{1/2} = 1.089778$$

$$\cos i = 0.419452$$

$$\eta = 2\pi/P \quad (\text{mean angular velocity})$$

$$\eta_t = 6.283185/6.087 \times 10^3 \text{ sec} = 1.032230 \times 10^{-3} \text{ rad/sec.}$$

4th - Computation of Ω_t , ω_t , and v_t

$$\Omega_t = 105^\circ 771 - 0.390 - 0.000 = 105^\circ 381$$

= $105^\circ 22' 52''$ (see formula in 2nd step above)

$$\omega_t = 28^\circ 876 - 0.049 = 28^\circ 827$$

= $28^\circ 49' 37''$

$$\tan 1/2 v_t = \left[\frac{1+e}{1-e} \right]^{1/2} \tan 1/2 E_t$$

where:

$$E_t = M_t + e_t \sin M_t + e_t \frac{e_t^2}{2} \sin 2 M_t$$

$$M_t = \eta_t (t-T)$$

If $\omega_t = 28^\circ 49' 37''$, then perigee must be $28^\circ 827$ beyond the ascending node because ω is the angular distance of perigee from the ascending node in the direction the satellite is moving.

If $P_t = 101.45$ minutes, the satellite travels through a mean angular distance of $3^\circ 549/\text{min.}$

The time (T_N) for the passage through the ascending node is Aug. 25.0530313; then the time T for the passage through perigee is:

$$T = T_N + \frac{28.827}{3.549 \times 60 \times 24} \text{ (in days)}$$

$$= \text{Aug 25.0586719}$$

Then:

$$t - T = 0.018782 \text{ days}$$

$$M_t = 1.032230 \times 10^{-3} \times 0.018782 \times 24 \times 3600$$

$$= 1.675066 \text{ radians or } 95^\circ 58' 27'' \text{ (mean anomaly)}$$

$$\sin M_t = 0.994569 \quad 2 M_t = 191^\circ 56' 54''$$

$$e_t \sin M_t = 0.085297 \quad \sin 2 M_t = -0.207030$$

$$e_t \frac{e_t^2}{2} \sin 2 M_t = -0.000761$$

$$E_t = 1.675066 + 0.085297 - 0.000761$$

$$= 1.759602 \text{ radians or } 100^\circ 49' 04'' \text{ (eccentric anomaly)}$$

$$1/2 E_t = 50^\circ 24' 32''$$

$$\tan 1/2 E_t = 1.209174$$

$$\tan 1/2 v_t = 1.317731$$

$$1/2 v_t = 52^\circ 48' 21''$$

$$v_t = 105^\circ 36' 42'' \text{ (true anomaly)}$$

5th - The computation of R may be included here since it is used in phase III

$$R_t = a (1 - e_t \cos E_t)$$

$$\cos E_t = -0.187686$$

$$e_t \cos E_t = -0.016097$$

$$1 - e_t \cos E_t = 1.016097$$

$$R_t = 1.146815 \text{ earth radii}$$

6th - Computation of δ and α

$$\sin \delta = \sin i \sin (\omega_t + v_t)$$

$$\omega_t + v_t = 134^\circ 26' 19''$$

$$\sin (\omega_t + v_t) = 0.714001$$

$\sin \delta = 0.648154$ (a negative $\sin \delta$ would indicate a position below the celestial equator)

$$\delta = 40^\circ 24' 09''$$

$$\alpha = \Omega_t + \text{arc tan} [\cos i \tan (\omega_t + v_t)]$$

$$\tan (\omega_t + v_t) = -1.019791$$

$$\cos i \tan (\omega_t + v_t) = -0.427753$$

$$\alpha = 105^\circ 22' 52'' + 156^\circ 50' 28''$$

$$\alpha = 262^\circ 13' 20'' \text{ or } 17^\text{h} 28^\text{m} 53.33$$

THE DETERMINATION OF RIGHT ASCENSION AND DECLINATION OF AN ARTIFICIAL EARTH SATELLITE FROM PHOTOGRAPHIC OBSERVATION

A photographic observation of a satellite should consist of a glass plate showing the image of the satellite against a field of stars and the record of the time corresponding to the instant of exposure. It is preferable that the exposure show the satellite and stars as points and not as trails. However, only the high precision instruments are capable of stopping the motion of the satellite; most field equipment is likely to produce a plate showing the stars as trails as well as the satellite. It should be remembered in evaluation of results that point images produce better accuracy than trails by a ratio of 10 to 1.

Identification and Selection of Reference Stars

At the time of the photographic observation, some record of the direction and the general position of the path of the satellite must be made in order that the reference stars can be identified. If it can be noted that the satellite was traveling through a certain constellation or close to a readily identifiable group of stars at the time of exposure, this information will be used to identify the stars which appear on the plate.

A more desirable method of recording the portion of the sky where the observation is being made, is to take a series of wide angle photographs which will include all of the exposures of the plates to be used in the reduction.

If star charts are available, they will be very helpful in identifying the stars on the plates. The charts together with star catalogs can be used to compare relative positions and work out the identifications.

If star charts are not available, a small graph of the area of the photograph can be made and the positions of the stars plotted from their coordinates in the catalog.

A minimum of three identified stars on the plate is needed to obtain a solution for the position of an unknown point on the plate, however, the solution is improved by a greater number of reference stars. The recommended number of identified stars is ten. The stars should be selected so that they are evenly distributed over the plate. They should also be selected so that the image of the satellite falls within the field of the reference stars. The solution is still possible if the unknown falls outside the field, but the results may be less accurate.

The best stars for the purpose are ones which produce small and clearly defined points of light. These are easier to measure than large stars which show up as irregular areas of varying density. The stars of magnitude 1 through 5 are nearly always too large to measure accurately. This depends on the length of exposure of the plate, but generally the stars of 6th or 7th magnitudes provide the best reference points.

Reduction to Epoch of the Coordinates of the Reference Stars

From the star catalogs, find and record the right ascension (α) and the declination (δ) of each reference star. These values must be corrected for epoch and proper motion.

For a star given in the Boss catalogs, epoch 1950, the following formulae will give the correction to mean place for the year of observation.¹²

$$\alpha_o = \alpha_m + (t_o - 1950) \text{ An. Var.} + 1/2 (t_o - 1950)^2 1/100 \text{ Sec. Var.} + \left[\frac{t_o - 1950}{100} \right]^3 3^d t$$

$$\delta_o = \delta_m + (t_o - 1950) \text{ An. Var.} + 1/2 (t_o - 1950)^2 1/100 \text{ Sec. Var.} + \left[\frac{t_o - 1950}{100} \right]^3 3^d t$$

Where α_m and δ_m are the values given for the beginning of the epoch, and t_o is the date of the exposure in terms of the year. For an observation done after 1950, the term $(t_o - 1950)$ will be positive.

Sample Computation

Correction to mean place for the year 1958 of star number 19320 from the Boss Catalog. The Right Ascension, 1950 is $14^h 16^m 52^s.427$. The corrections per year are: Annual Variation, + 2.4646; Secular Variation, - 0.0040; and the third term, + 0.008.

$$\begin{array}{rcl}
 \alpha_m & = & 14^h 16^m 52^s.427 \\
 (t_o - 1950) \text{ An. Var.} & = (8)(+2.4646) & + 19.7168 \\
 1/2 (t_o - 1950)^2 1/100 \text{ Sec. Var.} & = (0.32)(-0.0040) & - 0.0013 \\
 \left[\frac{t_o - 1950}{100} \right]^3 3^d t & = (0.0000512)(+0.008) & .0000 \\
 \hline
 \alpha_o & = & 14^h 17^m 12.1425
 \end{array}$$

The Declination, 1950 is $+38^\circ 59' 47".29$. The Corrections per year are An. Var. - 16.549, Sec. Var. + 0.209, and $3^d t + 0.024$.

$$\begin{array}{rcl}
 \delta_m & = & 38^\circ 59' 47".29 \\
 (t_o - 1950) \text{ An. Var.} & = (8)(-16.549) & - 2' 12.392 \\
 1/2 (t_o - 1950)^2 1/100 \text{ Sec. Var.} & = (0.32)(+0.209) & + .067 \\
 \left[\frac{t_o - 1950}{100} \right]^3 3^d t & = (0.0000512)(+0.024) & + .000 \\
 \hline
 \delta_o & = & 38^\circ 57' 34.965
 \end{array}$$

Additional corrections using the Besselian star numbers give the coordinates of the reference stars for the observation time. An ephemeris for the year of the observation is necessary in order to obtain the constants for the time of observation. The American Ephemeris and Nautical Almanac for 1958 gives the reduction formulas on page 246. The Besselian numbers, the capital letters in the formulae, are obtained from the table for the day and decimal part of the day for the observation. Bessel's Star Constants, the lower case letters, are determined from the following:

$$a = 3^\circ 07342 + 1.33613 \sin \alpha_o \tan \delta_o \quad a' = 20''0419 \cos \alpha_o$$

$$b = \frac{1}{15} \cos \alpha_o \tan \delta_o \quad b' = -\sin \alpha_o$$

$$c = \frac{1}{15} \cos \alpha_o \sec \delta_o$$

$$c' = \tan \epsilon \cos \delta_o - \sin \alpha_o \sin \delta_o$$

$$d = \frac{1}{15} \sin \alpha_o \sec \delta_o$$

$$d' = \cos \alpha_o \sin \delta_o$$

The formulas for reduction to apparent position are:

$$\alpha = \alpha_o + \tau \mu + Aa + Bb + Cc + Dd + E \text{ (in time)}$$

$$\delta = \delta_o + \tau \mu' + Aa' + Bb' + Cc' + Dd' \text{ (in arc)}$$

The formulas for reduction to apparent position by using the Independent Star Numbers are

$$\alpha = \alpha_o + f + \frac{1}{15} g \sin(G + \alpha_o) \tan \delta_o + \frac{1}{15} h \sin(H + \alpha_o) \sec \delta_o \text{ (in time)}$$

$$\delta = \delta_o + \tau \mu + g \cos(G + \alpha_o) + h \cos(H + \alpha_o) \sin \delta_o + i \cos \delta_o \text{ (in arc)}$$

where:

$$f + f' = +46.^{\text{m}}1015 A + 15 E = +46.^{\text{m}}1015 \tau + 0.917445 \delta \psi \text{ (in arc)}$$

$$= 3.^{\text{s}}07344 A + E = +3.^{\text{s}}07344 \tau + 0.0611630 \delta \psi \text{ (in time)}$$

$$f' = 3.^{\text{s}}073 A_2 = 0.06116 d \psi \text{ (in time)}$$

$$g \sin G = B$$

$$g \cos G = 20.^{\text{m}}0418 A$$

$$h \sin H = C$$

$$h \cos H = D$$

$$i = C \tan \epsilon$$

where:

τ = the time in units of one year from the beginning of the Besselian solar year. This can be determined from the table of Independent Star Numbers.

μ and μ' = the proper motion in right ascension and declination respectively obtained from the star catalogs.

α_o and δ_o = the values computed from the formulae converted to the year of observation. The numerical data in both of these reduction methods are for 1958 observations. Every ephemeris contains these conversion formulae for the particular year.

A_2 = terms of short period in A.

Sample Computation by Independent Star Numbers

a_o =	14 ^h 17 ^m 12 ^s 1425
f =	+ 2.499
f' =	+ 0.008
$\tau\mu$ =	+ 0.0002
$\frac{1}{15} g \sin (G + a_o) \tan \delta_o$ =	- 0.8612
$\frac{1}{15} h \sin (H + a_o) \sec \delta_o$ =	- 0.6935
	14 ^h 17 ^m 13 ^s 0950
δ_o =	38° 57' 34"965
$\tau\mu'$ =	+ 00.0155
$g \cos (G + a_o)$ =	- 08.9042
$h \cos (H + a_o) \sin \delta_o$ =	+ 10.9322
$i \cos \delta_o$ =	+ 05.5520
	38° 57' 42"5605

Determination of the Plate Center

The right ascension and declination of the center of the plate will be designated as A_o and D_o . There are three methods offered for finding the plate center, and the selection can be made on the basis of which method best fits the particular conditions.

Star Method

A star near the center of the plate may be used. If there is an identifiable star with a small and well defined image which appears near the center of the plate, the right ascension and declination of this star, corrected for epoch and proper motion as explained in the preceding section can be used for the A_o and D_o of the plate.

Displacing the center from the true geometrical center of the plate by a small amount doesn't cause any ill effects in the calculation.

Centroid Method

This method produces the center of the field of the reference stars which may or may not be the geometrical center of the plate. After the right ascension and declination for each star are found, the center may be calculated by taking the averages in each direction.

This can be written:

$$A_o = \frac{\sum a_i}{n}$$

$$D_o = \frac{\sum \delta_i}{n}$$

where Σ denotes the summations for $i = 1, 2, 3, \dots, n$.

Proportion Method

Another way to find the center of the plate is to pick the geometrical center G on the plate, and measure this location when the reference stars are measured. By computing a ratio of the proportional differences between two reference stars, find the A_o and D_o values as follows. (See Fig. 4)

- The values of a, b, and c are measured on the plate. (Line distances in millimeters.) G is the geometrical center of the plate. The coordinates of G are given as the differences in right ascension and declination between star number 5 and star number 4. Find α_o and δ_o as follows:

$$\alpha_o = \left(\frac{a}{a+b} \right) (\alpha_5 - \alpha_4) \quad (ck) \quad (1/15 \sec \delta_o \text{ estimated})$$

$$\delta_o = \left(\frac{a}{a+b} \right) (\delta_5 - \delta_4)$$

where

$\frac{a}{a+b}$ is the line ratio in mm. between stars 5 and 4.

c is the horizontal distance from the plate center (α_o, δ_o) to the line connecting stars 5 and 4.

δ_o estimated is the value of δ_o as read from the plate. (In practice a graph is made, keyed to the plate showing information given on Fig. 4).

k is a constant relationship of 1 revolution on the measuring engine or 1 mm. on the plate in seconds of arc.

$\alpha_5 - \alpha_4$ is the difference in right ascension between stars 5 and 4.

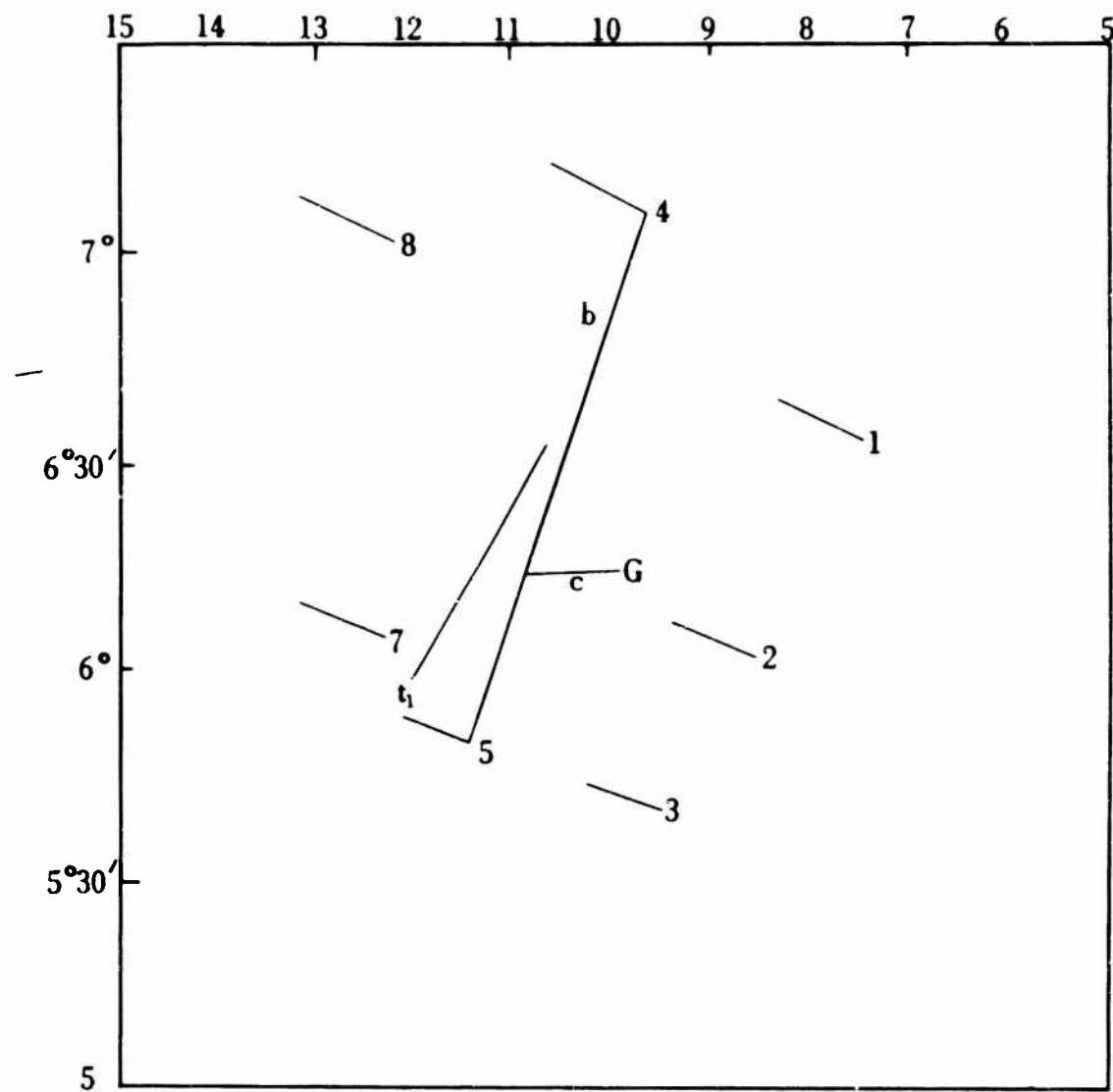
$\delta_5 - \delta_4$ is the difference in declination between stars 5 and 4.

After α_o and δ_o values are computed, we find A_o and D_o .

$$A_o = \alpha_5 - \alpha_o$$

$$D_o = \delta_5 - \delta_o$$

The A_o and D_o values should be rounded off to the nearest 10^5 for A_o and the nearest $1'$ for D_o .



Proportion Method for Finding the Plate Center

Fig. 4

Plate Measurements

A measuring engine of good quality is necessary for obtaining the plate measurements of the reference stars, the satellite, and the geometrical center, when the proportional method is used for arriving at the plate center. The measuring engine should have an accidental error of not greater than 5 microns for each screw and a systematic error of 0.25 micron.

The plate should be oriented in the engine so that the positive x direction corresponds with decreasing right ascension and the positive y direction corresponds with decreasing declination. In other words, the star with the largest right ascension, should have the smallest millimeter measurement for x, and the star farthest north or having the largest angle of declination should have the smallest y reading.

The measurements for the center of the plate depend upon which system is selected. If a star is selected to be substituted for the center, the measurements for its coordinates must be found along with those of all the reference stars. When the centroid method is used there will not be any measurements for the center. When the proportional method is selected, the measurements must be obtained as described in the previous section.

To obtain the x and y coordinates, the center reading must be subtracted from each corresponding reading of the stars and satellite. Where X_n , Y_n represent the readings of the stars and X_o , Y_o represent the readings of the center by any method, we will find the coordinates x_n , y_n by the following relationship:

$$x_n = X_n - X_o$$

$$y_n = Y_n - Y_o$$

When the centroid method is used to compute the coordinates of the center, the notation for the center is x_o , y_o . This is essentially translating the origin of the x, y axes to (0,0).

Computation of the Standard Coordinates of the Reference Stars

In finding the relationship between the right ascension and declination of stars and their rectangular coordinates on a photographic plate, it is necessary to employ the use of a set of coordinates which provide the transition between the celestial coordinates and the coordinates from the plate measurements. These transition coordinates actually lie in a plane which is tangent to the celestial sphere and is parallel to the camera plate. The geometrical relations and derivations will be given in Appendix D, however, here it is only necessary to know that these coordinates are referred to as the standard coordinates, ξ and η .

The standard coordinates are computed for each reference star by using the following formulae:⁴

$$\xi = \frac{\cot \delta \sin (\alpha - A_0)}{\sin D_0 + \cos D_0 \cot \delta \cos (\alpha - A_0)}$$

$$\eta = \frac{\cos D_0 - \cot \delta \sin D_0 \cos (\alpha - A_0)}{\sin D_0 + \cos D_0 \cot \delta \cos (\alpha - A_0)}$$

These values can be checked by substituting ξ and η of each star in the equation which will be presented in the final sections.

Computation of the Standard Coordinates of the Unknown Point

The next step is to find the standard coordinates of the unknown point. If an observation could be completely free from error, the plate coordinates would be identical with the standard coordinates. Since an observation is not without error, the discrepancies must be considered in order to make use of the observational data. The errors inherent in the x and y coordinates are due to refraction, aberration, and mechanical errors caused by the camera and measuring engine. (A more specific discussion of the errors appears in Appendix D.) It has been found that these errors have a linear effect on the plate coordinates, and the relationship between the plate coordinates and the standard coordinates of any point can be expressed:⁴

$$\xi - x = ax + by + c$$

$$\eta - y = dx + ey + f$$

In these equations, a,b,c,d,e,f, are small quantities which depend on the errors involved. They are called the plate constants.

In general, x differs from ξ and y differs from η by small quantities, and we can write these equations in an alternate form without loss of accuracy.

$$\xi - x = \xi a + \eta b + c$$

$$\eta - y = \xi d + \eta e + f$$

Theoretically, the plate constants can be computed from either set of equations, but since the ξ and η values can be computed more accurately than the x and y values can be measured, it is recommended that equations using the standard coordinates instead of the plate coordinates be used to determine the plate constants.

Two methods for solution are included here.

The first method is to obtain the specific values of the six plate constants. We can write three equations with three unknowns which can be solved by determinants or by any method for solving simultaneous equations.

Using the form of equation with the standard coordinates we can write n equations, where n is the number of reference stars.

$$\xi_1 - x_1 = a\xi_1 + b\eta_1 + c$$

$$\xi_2 - x_2 = a\xi_2 + b\eta_2 + c$$

.....

$$\xi_n - x_n = a\xi_n + b\eta_n + c$$

These are called condition equations from which we can form the normal equations. The procedure is as follows:

First we multiply each ξ and η by the focal length of the camera in millimeters. It was previously stated that ξ differs from x and η from y by small amounts. When ξ and η are multiplied by the focal length, it is noticeable that the differences are small. We then set up a schematic arrangement as shown below:

where $s_i = \xi_i + \eta_i + l$

and $m_i = \xi_i - x_i$

coefficients

<u>of a</u>	<u>of b</u>	<u>of c</u>	<u>s</u>	<u>m</u>
ξ_1	η_1	1	s_1	m_1
ξ_2	η_2	1	s_2	m_2
.
.
.
ξ_n	η_n	1	s_n	m_n

First, take each ξ and multiply it by each quantity in its row. Then find the sum of each column.

$\xi_1 \xi_1$	$\xi_1 \eta_1$	ξ_1	$\xi_1 s_1$	$\xi_1 m_1$
$\xi_2 \xi_2$	$\xi_2 \eta_2$	ξ_2	$\xi_2 s_2$	$\xi_2 m_2$
.
.
.
$\xi_n \xi_n$	$\xi_n \eta_n$	ξ_n	$\xi_n s_n$	$\xi_n m_n$
$\underline{[\xi_i \xi_i]}$	$\underline{[\xi_i \eta_i]}$	$\underline{[\xi_i]}$	$\underline{[\xi_i s]}$	$\underline{[\xi_i m]}$

The square brackets denote summations for $i = 1, 2, 3, \dots, n$.

In the same way, proceed with η to produce

$$[\eta, \eta_i], [\eta_i], [n_i s_i], \text{ and } [n_i m_i]$$

Also find

$$[s_i m_i] \text{ and } [m_i]$$

If these computations are correct, we should have

$$[\xi_i s_i] = [\xi_i \xi_i] + [\xi_i \eta_i] + [\xi_i]$$

$$[\eta_i s_i] = [\xi_i \eta_i] + [\eta_i \eta_i] + [\eta_i]$$

$$[s_i] = [\xi_i] + [\eta_i] + [m_i]$$

$$[s_i m_i] = [\xi_i m_i] + [\eta_i m_i] + [m_i]$$

If these checks all work out satisfactorily, then we can go ahead and form the normal equations as follows:

$$[\xi_i \xi_i]a + [\xi_i \eta_i]b + [\xi_i]c = [\xi_i(\xi_i - x_i)]$$

$$[\xi_i \eta_i]a + [\eta_i \eta_i]b + [\eta_i]c = [\eta_i(\xi_i - x_i)]$$

$$[\xi_i]a + [\eta_i]b + nc = [(\xi_i - x_i)]$$

These equations can now be solved for a, b, and c.

The procedure for finding d, e, and f is the same. The only difference in setting up the schematic arrangement is that

$$m_i = \eta_i - y_i$$

The resulting normal equations in d, e, and f are:

$$[\xi_i \xi_i]d + [\xi_i \eta_i]e + [\xi_i]f = [\xi_i(\eta_i - y_i)]$$

$$[\xi_i \eta_i]d + [\eta_i \eta_i]e + [\eta_i]f = [\eta_i(\eta_i - y_i)]$$

$$[\xi_i]d + [\eta_i]e + nf = [(\eta_i - y_i)]$$

To check the computations of the plate constants, substitute the values back into condition equations.

After the values of the plate constants have been found, the standard coordinates ξ_t and η_t of the unknown point can be computed by substituting in either pair of equations which were used to find the plate constants.

For the same reason it was recommended to use the standard coordinates for finding the plate constants, it is recommended that the following equations be used to find the standard coordinates of the unknown point.

$$\xi_t - x_t = a\xi_t + b\eta_t + c$$

$$\eta_t - y_t = d\xi_t + e\eta_t + f$$

In these equations, since both unknowns appear in each equation, we must solve them as simultaneous equations.

ξ_t and η_t must now be divided by the focal length for use in the final equations which are presented in the section entitled Celestial Coordinates of the Satellite.

Another calculation for obtaining the standard coordinates is called the method of dependences.¹¹ Instead of solving for the specific values of a, b, ..., f, we compute certain quantities which depend on the reference stars selected and on one position of the satellite. The development of this method will be more fully discussed in Appendix D.

The terms a, b, ..., f are replaced by P, Q, and R. The resulting equations which we can now solve are:

$$P[x_i x_i] + Q[x_i y_i] + R[y_i] = x_t$$

$$P[x_i y_i] + Q[y_i y_i] + R[x_i] = y_t$$

$$P[x_i] + Q[y_i] + Rn = 1$$

When the centroid method of obtaining the plate center is used, these three equations become:

$$P[x_i x_i] + Q[x_i y_i] = x_t$$

$$P[x_i y_i] + Q[y_i y_i] = y_t$$

$$Rn = 1$$

The fact that the sum of the x coordinates and the sum of the y coordinates are zero provides a useful check on the x and y coordinates which are computed when the plate center is the centroid.

After P, Q, and R are determined, the dependences of the reference stars are formed by means of

$$n_i = Px_i + Qy_i + R$$

A check on this calculation is

$$\sum_{i=1}^n D_i = 1$$

Now that the dependences have been found we can compute the standard coordinates of the unknown point.

For ξ_t

$$\xi_t - x_t = \sum_{i=1}^n D_i (\xi_i - x_i)$$

and for η_t

$$\eta_t - y_t = \sum_{i=1}^n D_i (\eta_i - y_i)$$

In this computation, just as in the first solution, the focal length must be considered. The ξ and η coordinates of the reference stars can be multiplied by the focal length (in millimeters) or the x and y coordinates of the reference stars and the satellite can be divided by the focal length. If the second alternative is chosen, ξ_t and η_t will be in the proper units to substitute directly in the formulae for finding the standard coordinates of the unknown point. If the first alternative is selected, ξ_t and η_t must be divided by the focal length before using them in the formulae.

This method of dependences is best suited to the type of observation where a series of plates is taken of the same portion of the sky; i.e., that the plates all have the same center, and the position of the unknown point changes very little between plates.

Celestial Coordinates of the Satellite

The standard coordinates ξ_t and η_t determined from either of the two methods are now used to find the right ascension (α_t) declination (δ_t) of the satellite.

First we find α_t

$$\tan(\alpha_t - A_o) = \frac{\xi_t \sec D_o}{1 - \eta_t \tan D_o}$$

Then using the values of α_t from this equation, we can find δ_t from:

$$\cot \delta_t \cos(\alpha_t - A_o) = \frac{1 - \eta_t \tan D_o}{\eta_t + \tan D_o}$$

or

$$\cot \delta_t \sin(\alpha_t - A_o) = \frac{\xi_t \sec D_o}{\eta_t + \tan D_o}$$

Procedure

The given information for the computation is the right ascension (α) and declination (δ) of each reference star, the plate measurements of the reference stars and the unknown point, and the focal length of the camera.

The computation procedure is as follows:

After the identification and selection of the reference stars, their coordinates must be corrected to the time of observation. This can be done by using either the Besselian Star Constants or the Independent Star Numbers. The choice of the latter method is recommended in the reduction of satellite plates. If there were many observations on different dates of the same group of stars, the use of the Besselian Star Constants would have the advantage, but usually each satellite plate includes a different group of reference stars, and so the Independent Star Numbers method is better suited to the computation.

The second step is to find the astronomical coordinates (A_o, D_o) of the center of the photographic plate by the Centroid Method. The equations are:

$$A_o = \frac{\sum \alpha_i}{n}$$

$$D_o = \frac{\sum \delta_i}{n}$$

Third, find the plate coordinates (x_o, y_o) of the center of the plate by the Centroid Method also. For the plate coordinates this can be written

$$x_o = \frac{\sum X_i}{n}$$

$$y_o = \frac{\sum Y_i}{n}$$

The capital letters X and Y refer to the readings just as they are taken from the plate. The lower case letters x and y are the plate coordinates which have been referred to the plate center. Do not include the measurements of the unknown point X_t and Y_t in this step.

The fourth step is to find the plate coordinates of the reference stars. Using the

plate center found in step three, subtract the x coordinate from each X reading; and the y coordinate of the center from each Y reading.

$$x_n = X_n - x_o$$

$$y_n = Y_n - y_o$$

This step can be checked by the relationship

$$\sum x_n = \sum y_n = 0$$

Find the plate coordinates of the unknown point

$$x_t = X_t - x_o$$

$$y_t = Y_t - y_o$$

The fifth step is to compute the standard coordinates ξ and η . Note that the denominators are the same in the two equations. Declination is given in degrees, minutes, and seconds of arc. Right Ascension is given in hours, minutes and seconds of time, and must be converted to arc by using the following relationship:

$$1^h = 15^\circ, 1^m = 15', 1^s = 15''$$

After the standard coordinates have been computed, multiply each ξ and η by the focal length in millimeters.

Sixth, find the plate constants a, b, c, d, e, f. It is recommended that the standard coordinates be used for this solution rather than the plate coordinates.

$$\xi - x = \xi a + \eta b + c$$

$$\eta - y = \xi d + \eta e + f$$

The solution for a, b, and c is carried out by following the least square solution described in the section entitled "Computation of the Standard Coordinates of the Unknown Point". Then the equations for d, e, and f are solved by repeating the procedure.

The standard coordinates for the unknown point are then found by making use of the plate constants from step six.

$$\xi_t - x_t = \xi_t a + \eta_t b + c$$

$$\eta_t - y_t = \xi_t d + \eta_t e + f$$

These equations have to be solved simultaneously for ξ_t and η_t . When ξ_t and η_t are divided by the focal length they can be substituted in the same equations used in step five to find the right ascension and declination of the unknown point.

Sample Computation

Coordinates, corrected to epoch, of the reference stars

<u>Star</u>	<u>Boss Catalog Number</u>	<u>α</u>	<u>δ</u>
1	19429	14 ^h 22 ^m 02 ^s .2994	+ 41° 02' 09".1446
2	19320	14 17 13.0946	+ 38 57 42.5604
3	19225	14 12 42.7228	+ 41 43 01.8724
4	19124	14 08 17.2595	+ 40 58 38.4310
5	19414	14 21 49.9457	+ 39 31 17.4283
6	19322	14 17 17.4187	+ 42 12 01.9742

Plate Measurements

<u>Star</u>	<u>X</u>	<u>Y</u>
1	60.4910	45.7711
2	69.6317	54.0542
3	67.6740	38.4797
4	73.4747	40.1880
5	63.9537	53.2307
6	62.3659	38.0226

Satellite

<u>X_t</u>	<u>Y_t</u>
68.6078	48.6072

Focal length of camera

311.66 mm

Astro coordinates of the center of the field of stars

$$A_o = \frac{85^h 39^m 22.7407}{6} = 14^h 16^m 33.7901$$

$$D_o = \frac{244^\circ 24' 51".4109}{6} = 40^\circ 44' 8".5684$$

Plate coordinates of the center of the field of stars.

$$x_o = \frac{397.5910}{6} = 66.2652$$

$$y_o = \frac{269.7463}{6} = 44.9577$$

Plate coordinates of the reference stars.

<u>Star</u>	<u>$x_n = X_n - x_o$</u>	<u>$y_n = Y_n - y_o$</u>
1	- 5.7742	+ 0.8134
2	+ 3.3665	+ 9.0965
3	+ 1.4088	- 6.4780
4	+ 7.2095	- 4.7697
5	- 2.3115	+ 8.2730
6	- 3.8993	- 6.9351

Plate coordinates of the satellite

$$\begin{array}{l} \xleftarrow{x_t = X_t - x_o} \\ +2.3426 \end{array} \quad \begin{array}{l} \xleftarrow{y_t = Y_t - y_o} \\ +3.6495 \end{array}$$

Computing the Standard Coordinates of each reference star.

<u>$\cot \delta$</u>	<u>$\sin (\alpha - A_o)$</u>	<u>$\cos (\alpha - A_o)$</u>
+ 1.148914781	+ 0.023887597	+ 0.999714651
+ 1.236580996	+ 0.002858299	+ 0.999995915
+ 1.121697783	- 0.016802899	+ 0.999858821
+ 1.151287611	- 0.036100879	+ 0.999348151
+ 1.212169628	+ 0.022989457	+ 0.999735708
+ 1.102825132	+ 0.003172755	+ 0.999994967

<u>$\sin D_o$</u>	<u>$\cos D_o$</u>
+ 0.652570836	+ 0.757727725
<u>$(311.66)(\xi)$</u>	<u>$(311.66)(\eta)$</u>
+ 5.616602180	+ 1.676782734
+ 0.693002317	- 9.651529777
- 3.909829081	+ 5.361103341
- 8.497551116	+ 1.414961445
+ 5.528992968	- 6.565193637
+ 0.732758139	+ 7.970477097

Condition equations for finding the plate constants a, b, and c:

$$\begin{aligned} 5.616602180 a + 1.676782734 b + c &= 11.390762180 \\ 0.693002317 a - 9.651529777 b + c &= - 2.673527683 \\ - 3.909829081 a + 5.361103341 b + c &= - 5.318659081 \\ - 8.497551116 a + 1.414961445 b + c &= - 15.707081116 \\ 5.528992968 a - 6.565193637 b + c &= 7.840452968 \\ 0.732758139 a + 7.970477097 b + c &= 4.632018139 \end{aligned}$$

$$\begin{aligned}
[\xi_1 \xi_1] &= 150.628308411 & [\eta_1 (\eta_1 - y_1)] &= 470.836618882 \\
[\xi_1 - x_1] &= 0.163965407 & [\xi_1] &= 0.163975407 \\
[\eta_1 (\xi_1 - x_1)] &= -20.390027237 & [\xi_1 (\xi_1 - x_1)] &= 263.135349755 \\
[\xi_1 \eta_1] &= -60.713893397 & [\eta_1] &= 0.206601203 \\
[\eta_1 \eta_1] &= 233.337444947 & [\eta_1 - y_1] &= 0.206591203 \\
[\xi_1 (\eta_1 - y_1)] &= -178.104463367
\end{aligned}$$

Normal equations for finding a, b, and c:

$$\begin{aligned}
150.628308411 a - 60.713893397 b + 0.163975407 c &= 263.135349755 \\
-60.713893397 a + 233.337444947 b + 0.206601203 c &= -20.390027237 \\
0.163975407 a + 0.206601203 b + 6 c &= 0.163965407
\end{aligned}$$

Solving these equations by determinants:

$$\begin{aligned}
a &= 1.912312557 \\
b &= 0.410229843 \\
c &= -0.039060134
\end{aligned}$$

Condition equations for finding the plate constants d, e, and f:

$$\begin{aligned}
5.616602180 d + 1.676782734 e + f &= 0.863382734 \\
0.693002317 d - 9.651529777 e + f &= -18.748029777 \\
-3.909829081 d + 5.361103341 e + f &= 11.839103341 \\
-8.497551116 d + 1.414961445 e + f &= 6.184661445 \\
5.528992968 d - 6.565193637 e + f &= -14.838193637 \\
0.732758139 d + 7.970477097 e + f &= 14.905577097
\end{aligned}$$

Normal equations for finding d, e, and f:

$$\begin{aligned}
150.628308411 d - 60.713893397 e + 0.163975407 f &= -178.104463367 \\
-60.713893397 d + 233.337444947 e + 0.206601203 f &= 470.836618882 \\
0.163975407 d + 0.206601203 e + 6 f &= 0.206591203
\end{aligned}$$

Solving by determinants:

$$\begin{aligned}
d &= -0.412290640 \\
e &= 1.910576330 \\
f &= -0.020088440
\end{aligned}$$

Standard coordinates of the satellite:

$$\begin{aligned}
1.912312557 \xi_t + 0.410229843 \eta_t - 0.039060134 &= \xi_t - 2.3426 \\
-0.412290640 \xi_t + 1.910576330 \eta_t - 0.020088440 &= \eta_t - 3.6495 \\
\xi_t &= -0.608774171 \\
\eta_t &= -4.261458733
\end{aligned}$$

Right ascension and declination of the satellite:

$$\tan(\alpha_t - A_o) = \frac{(-0.608774171)(1.319735265)}{1 + (4.261458733)(0.861220745)}$$
$$= -0.002547872$$

$$\alpha_t = 14^h 15^m 58.754$$

$$\cot \delta_t \sin(\alpha_t - A_o) = -0.003041571$$

$$\delta_t = 39^\circ 57' 08.0689$$

THE DETERMINATION OF THE GEOCENTRIC COORDINATES OF A POSITION ON THE EARTH'S SURFACE FROM SATELLITE OBSERVATIONS

In the first phase of this report the procedure is presented for finding the coordinates of a satellite by computation using the elements of the orbit. The resulting coordinates have as their origin the center of the earth.

In phase two the method is given for finding the coordinates of a satellite from a photographic observation. The coordinates determined from observations are based on an origin located at the observation station.

The coordinates of the satellite which are determined from orbital data are those which would be seen by an observer at the center of the earth. These are called geocentric coordinates of the satellite. Those which are observed from the surface of the earth are called the observed or apparent positions.

The parallax of the satellite is the difference of directions of the straight lines drawn to the satellite from the two different reference points. It may also be defined as the angle at the satellite made by the line connecting the center of the earth and the satellite and the line connecting the observation station on the surface of the earth and the satellite.¹⁰

The observer at O in figure 5 views the satellite S along the line OS. When the center of the earth is the reference point, the position of the satellite is referred to line CS. The angle CSO is called the parallax of the satellite.

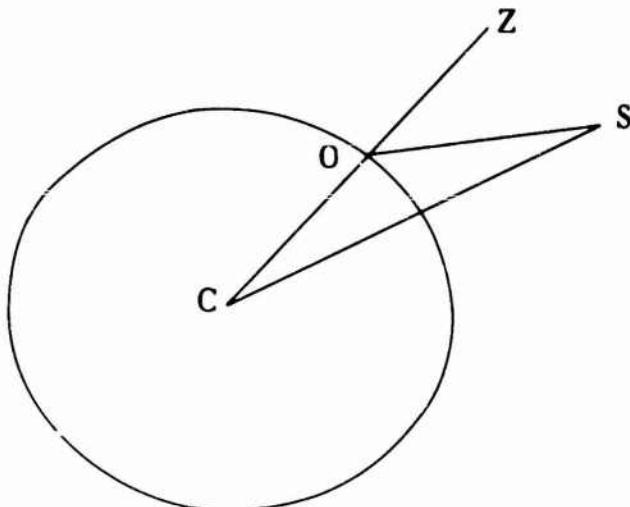


Fig. 5

The coordinates obtained from the photographic observation differ from the ones resulting from use of the orbital data. The discrepancy describes the difference in perspective between the earth centered location and the position on the earth's surface. If the satellite were precisely at the observer's zenith, there should be no discrepancy.

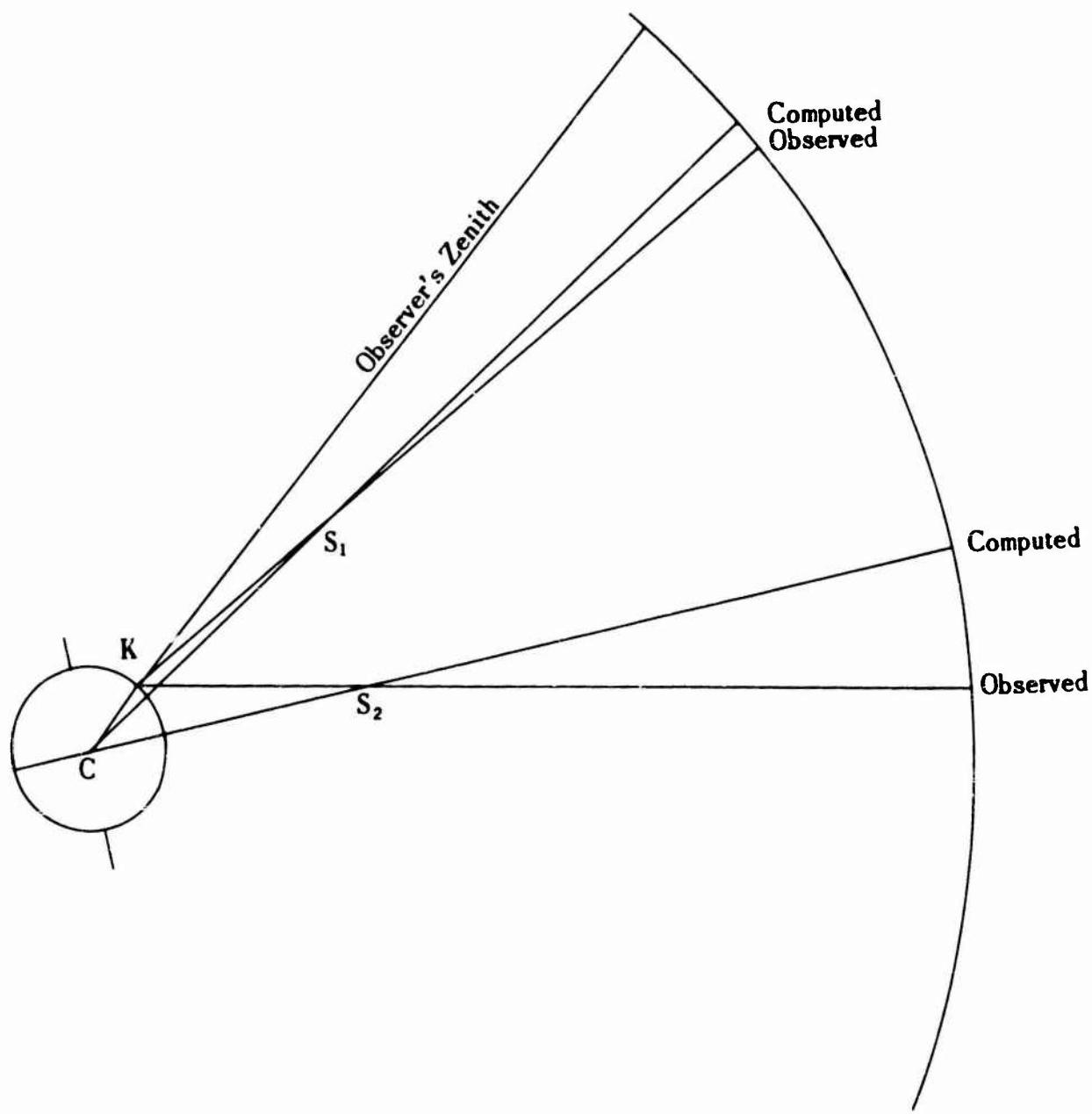


Fig. 6

between the observed coordinates and the computed geocentric coordinates, and it would not be possible to compute the observer's position from satellite observations.

The location of the satellite for the photographic observation is of considerable importance. The ideal location would be a small angular distance from the observer's zenith. Very incongruous results are obtained if a satellite is photographed low on the horizon.

Since the satellite is a great deal closer to the earth than any of the reference stars, the projection of the satellite's image onto the field of stars depends on the origin of the projection. Suppose the satellite is in position S in figure 6, the observer at K sees the satellite projected on the celestial sphere at the point marked observed. When the satellite isn't far from the observer's zenith the observed and computed positions are close together. The lower a satellite is toward the horizon, the more its observed position differs from the computed position which is referenced to the center of the earth.

It is actually the fact that there exists a difference between the earth-centered position and the observed position which allows us to compute the observer's position. However, the formulae depend on the parallax angle being small.

The present orbits of the satellites are undergoing considerable changes due to the perturbations already discussed. These changes in the orbit cause inaccuracies in the results of phase one which in turn affects phase two. Phase one is hindered if the times of the observations to be used are too widely separated or if there is much of a time lapse between the epoch of the furnished orbital information and the time of the observation. It is not recommended that observations be any farther apart than two consecutive orbits.

The accuracy loss which results from the time lapse between the furnished orbital data and the time of observation depends on each particular satellite. The orbital information for a satellite with a fairly steady orbit suffers less accuracy loss between revolutions than the satellite which is experiencing severe orbital disturbances.

Many other factors contribute to the loss of accuracy, but the need here is more to point out the fact that many different factors can contribute to errors in the results than to itemize the possible trouble spots. Careful selection of the input data for phase three is very essential to reasonable results.

Methods for Finding the Observer's Coordinates

Several methods exist for finding the position of an observation station by using some celestial object as a triangulation point.

The Moon Camera Method for finding the observer's position developed by Dr. Wm. Markowitz of the U. S. Naval Observatory is adaptable for use with an artificial earth satellite. The two methods which follow have been adapted to fit the situation which exists in the case of the satellite.¹⁴ The Linear Method provides a way to find the coordinates of an unknown station while the Differential Method results in corrections to an approximate position.

Linear Method

Each observation of the satellite yields the coordinates α' , δ' and the time from the photographic plate reduction phase; the coordinates α, δ and the value of R from computation based on orbital data. These six values provide the input for one observation. At least two observations are necessary in order to solve for the geocentric coordinates x_o, y_o, z_o of the observation station. The equations which relate all the observed and computed data are

$$x_o \sin G' + y_o \cos G' = R \cos \delta \sin (\alpha' - \alpha)$$
$$x_o \cos G' - y_o \sin G' - z_o \cot \delta' = R \cos \delta \cos (\alpha' - \alpha)$$
$$- R \sin \delta \cot \delta'$$

where:

$$G' = \alpha' - \gamma$$

γ = The hour angle of the Vernal Equinox with respect to Greenwich

R = The distance from the center of the earth to the satellite in units of earth radii.

In the numerical example included in this phase, there is an explanation of the treatments of the time of the observation including the determination of γ .

Since R must be used in the equations in units of earth radii, the values of x_o, y_o and z_o will be in the same units. To find their equivalent values in meters, multiply each by the semimajor axis (a) of the reference ellipsoid of the area. The numerical example which follows gives us the geocentric coordinates of a station in North America, and therefore, we multiply each coordinate by the a of the Clarke 1866 ellipsoid.

ELLIPSOID	SEMIMAJOR AXIS - a
Clarke 1866	6,378,206.4 Meters
International	6,378,388 Meters
Hough	6,378,270 Meters
Bessel	6,377,397.155 Meters

Differential Method

Again we need at least two observations to get a solution. The results to be obtained are small corrections to be applied to the space coordinates. First, we start with an approximate position. If the geodetic coordinates are the given information, the x, y, z space coordinates need to be computed; Appendix H gives the necessary information for that conversion. We denote these preliminary coordinates as x_c, y_c, z_c , and using them in the following equations, we find α_c^i and δ_c^i .

$$\tan(\alpha'_c - \alpha) = \frac{x_a \sin G + y_a \cos G}{-x_a \cos G + y_a \sin G + R \cos \delta}$$

$$\tan \delta'_c = \frac{\cos(\alpha'_c - \alpha)(R \sin \delta - z_a)}{-x_a \cos G + y_a \sin G + R \cos \delta}$$

These values just obtained and the observed values are combined to find $\Delta\alpha'$ and $\Delta\delta'$. Both of these increments should be small.

$$\alpha' - \alpha'_c = \Delta\alpha'$$

$$\delta' - \delta'_c = \Delta\delta'$$

As long as the increment of any of the coordinates is small, it can be considered the same as the differential. Therefore, we can relate all the observed data with the unknowns by two differential equations to be solved for Δx_o , Δy_o , Δz_o .

$$a_1 dx_o + a_2 dy_o = a_3 d\alpha'$$

$$b_1 dx_o + b_2 dy_o + b_3 dz_o = b_4 d\alpha' + b_5 d\delta'$$

where:

$$a_1 = \sin G + \tan(\alpha' - \alpha) \cos G$$

$$a_2 = \cos G - \tan(\alpha' - \alpha) \sin G$$

$$a_3 = \sec^2(\alpha' - \alpha)(R \cos \delta - x_a \cos G - \Delta x \cos G + y_a \sin G + \Delta y \sin G)$$

$$b_1 = \tan \delta' \cos G$$

$$b_2 = -\tan \delta' \sin G$$

$$b_3 = -\cos(\alpha' - \alpha)$$

$$b_4 = \sin(\alpha' - \alpha)(R \sin \delta - z_a - \Delta z)$$

$$b_5 = \sec^2 \delta' (R \cos \delta - x_a \cos G - \Delta x \cos G + y_a \sin G + \Delta y \sin G)$$

$$G = \alpha - \gamma$$

The solutions are then used to find the corrected values of the observer's coordinates.

$$x_o = x_a + \Delta x$$

$$y_o = y_a + \Delta y$$

$$z_o = z_a + \Delta z$$

Procedure

Linear Method

The given information consists of the right ascension (α), declination (δ), and the

radius vector (R) for each observation computed from orbital data together with the right ascension (α'), declination (δ') and the time of each observation computed from photographic plates.

Computation of γ :

The hour angle (γ) of the Vernal Equinox is computed from the time of the observation and from an ephemeris for the year. A description of the method is included in the sample computations which follow.

Computation of x_o , y_o , and z_o :

Tabulate the necessary factors to be used in the equations. Substitute the values into the equations, writing two equations for each observation. The linear equations can then be solved for x_o , y_o , and z_o . These coordinates may then be converted into latitude and longitude. Appendix H contains the equations for this conversion.

Differential Method

The given information is the same as that for the Linear Method with the addition of the approximate coordinates (x_a , y_a , z_a) of the observation site.

First we compute α'_c and δ'_c using the equations:

$$\tan(\alpha'_c - \alpha) = \frac{x_a \sin G + y_a \cos G}{-x_a \cos G + y_a \sin G + R \cos \delta}$$

$$\tan \delta'_c = \frac{\cos(\alpha'_c - \alpha) (R \sin \delta - z_a)}{-x_a \cos G + y_a \sin G + R \cos \delta}$$

Then, find the differences between these computed values and the observed values:

$$\Delta \alpha' = \alpha' - \alpha'_c$$

$$\Delta \delta' = \delta' - \delta'_c$$

Find the eight coefficients for the differential equations. After this step the differential equations can be formed and solved for the increments to add to the approximate coordinates.

Usually, several approximations will be necessary. When the increments are added to the approximate coordinates of the first computation, the resulting values can be used for the approximate coordinates for the second approximation. This can be repeated as many times as necessary until the increments become very small.

Sample Computation

Computation for γ - The hour angle of the Vernal Equinox:

The hour angle of the Vernal Equinox is sidereal time. In this report, our reference point is the Greenwich meridian, and so we start with the Universal Time (UT) or Greenwich Civil Time (GCT) of the observation.

If the UT of observation is 1958 August 25 at $1^h 51^m 31.98^s$, we must find the side-

real time at Greenwich. From The American Ephemeris and Nautical Almanac for 1958, we find that Greenwich sidereal time on August 24 at GCT 0^h (midnight) is 22^h 11^m 00^s.488. The solar interval of 1^h 51^m 31^s.98 must be converted to its equivalent in sidereal time. The sidereal day is 3^m 56^s.556 longer than the civil day; therefore a correction must be added to convert solar time to sidereal time. The almanac for any year provides tables for easy conversion.

The mean solar interval of 1^h 51^m 31^s.98 must be increased by 18^s.322, and the equivalent sidereal time is 1^h 51^m 50^s.302.

At GCT 0^h the sidereal time was 22^h 11^m 00^s.488. Hence the sidereal time of the observation is 24^h 02^m 50^s.79.

Linear Method

Given Information:

Observation One

$$\begin{aligned}\alpha &= 17^{\text{h}} 28^{\text{m}} 53\text{.}333 \\ \delta &= 40^{\circ} 24' 09'' \\ R &= 1.146815 \text{ e.r.} \\ \alpha' &= 14^{\text{h}} 15^{\text{m}} 58\text{.}75 \\ \delta' &= 39^{\circ} 57' 08''\text{.}07 \\ \gamma &= 24^{\text{h}} 02^{\text{m}} 50\text{.}79\end{aligned}$$

Observation Two

$$\begin{aligned}\alpha &= 6^{\text{h}} 06^{\text{m}} 33\text{.}8 \\ \delta &= 43^{\circ} 32' 42'' \\ R &= 1.126957 \text{ e.r.} \\ \alpha' &= 9^{\text{h}} 06^{\text{m}} 00\text{.}02 \\ \delta' &= 72^{\circ} 38' 24''\text{.}21 \\ \gamma &= 11^{\text{h}} 58^{\text{m}} 09\text{.}47\end{aligned}$$

Tabulation:

$$\begin{array}{llll}G' & = 14^{\text{h}} 13^{\text{m}} 07\text{.}96 & G' & = 2^{\text{h}} 52^{\text{m}} 09\text{.}45 \\ \sin G' & = -0.548777237 & \sin G' & = -0.682500802 \\ \cos G' & = -0.835968627 & \cos G' & = 0.730884844 \\ \sin \delta & = 0.648153129 & \sin \delta & = 0.688924071 \\ \cos \delta & = 0.761510027 & \cos \delta & = 0.724833515 \\ (\alpha' - \alpha) & = -3^{\text{h}} 12^{\text{m}} 54\text{.}583 & (\alpha' - \alpha) & = 2^{\text{h}} 59^{\text{m}} 27\text{.}02 \\ \sin (\alpha' - \alpha) & = -0.745795003 & \sin (\alpha' - \alpha) & = 0.705408843 \\ \cos (\alpha' - \alpha) & = 0.666175513 & \cos (\alpha' - \alpha) & = 0.708800652 \\ \cot \delta' & = 1.193772997 & \cot \delta' & = 0.312613361\end{array}$$

Equations:

$$\begin{aligned} -0.548777237 x_o - 0.835968627 y_o &= -0.651311071 \\ -0.835968627 x_o + 0.548777237 y_o - 1.193772997 z_o &= -0.305566988 \\ -0.682500802 x_o + 0.730884844 y_o &= 0.576217590 \\ 0.730884844 x_o + 0.682500802 y_o - 0.312613351 z_o &= 0.336279009 \end{aligned}$$

The solution of these equations by Doolittle's method yields:¹⁵

$$\begin{aligned} x_o &= 0.005833871 \text{ e.r.} \\ &= 37,209.63 \text{ meters} \\ y_o &= 0.782937022 \text{ e.r.} \\ &= 4993,733.93 \text{ meters} \\ z_o &= 0.619969093 \text{ e.r.} \\ &= 3954,290.84 \text{ meters} \end{aligned}$$

The geodetic coordinates of this point are:

$$\phi = 38^\circ 33' 45.78''$$

$$\lambda = 90^\circ 25' 36.91''$$

Differential Method

Given Information:

Observation One

$$\begin{aligned} \alpha &= 17^\circ 28^m 53.333^s \\ \delta &= 40^\circ 24' 09'' \\ R &= 1.146815 \text{ e.r.} \\ \alpha' &= 14^\circ 15^m 58.75^s \\ \delta' &= 39^\circ 57' 08.07'' \\ \gamma &= 24^\circ 02^m 50.79'' \end{aligned}$$

Observation Two

$$\begin{aligned} \alpha &= 6^\circ 06^m 33^s \\ \delta &= 43^\circ 32' 42'' \\ R &= 1.126957 \text{ e.r.} \\ \alpha' &= 9^\circ 06^m 00.02^s \\ \delta' &= 70^\circ 38' 24.21'' \\ \gamma &= 11^\circ 58^m 09.47'' \end{aligned}$$

$$x_o = -0.005750000$$

$$y_o = 0.789900000$$

$$z_o = 0.619950000$$

Tabulation:

G	= 17 ^h 26 ^m 02 ^s 543	G	= -5 ^h 51 ^m 36 ^s 47
sin G	= -0.989043181	sin G	= -0.999329646
cos G	= -0.147626508	cos G	= 0.036609552
cos δ	= 0.761510027	cos δ	= 0.724833515
sin δ	= 0.648153129	sin δ	= 0.688924071
(α' - α)	= -3 ^h 12 ^m 54 ^s 583	(α' - α)	= 2 ^h 59 ^m 27 ^s 02
sin (α' - α)	= -0.745795003	sin (α' - α)	= 0.705408843
cos (α' - α)	= 0.666175513	cos (α' - α)	= 0.708800652
tan (α' - α)	= -1.119517287	tan (α' - α)	= 0.99521472
sec ² (α' - α)	= 2.253318953	sec ² (α' - α)	= 1.99045235
tan δ'	= 0.83768021	tan δ'	= 3.19883964
sec ² δ'	= 1.70170814	sec ² δ'	= 11.23257504

Computation of Δα' and Δδ' for Observation One:

$$\tan (\alpha'_c - \alpha) = \frac{-0.110923180}{0.091217061}$$

$$= -1.216035458$$

$$\alpha'_c = 211^\circ 39' 15\overset{''}{.}128$$

$$\cos (\alpha'_c - \alpha) = 0.635161721$$

$$\tan \delta'_c = 0.858991162$$

$$\delta'_c = 40^\circ 39' 44\overset{''}{.}234$$

$$\alpha' - \alpha'_c = \Delta \alpha' = 2^\circ 20' 26\overset{''}{.}127$$

$$= 0.040851016 \text{ radians}$$

$$\delta' - \delta'_c = \Delta \delta' = -0^\circ 42' 36\overset{''}{.}164$$

$$= -0.012392632 \text{ radians}$$

Computation of $\Delta\alpha'$ and $\Delta\delta'$ for Observation Two:

$$\begin{aligned}\tan(\alpha'_c - \alpha) &= \frac{0.034664031}{0.027696221} \\ &= 1.251579789 \\ \alpha'_c &= 143^\circ 00' 46.755'' \\ \cos(\alpha'_c - \alpha) &= 0.624213892 \\ \tan \delta'_c &= 3.525775217 \\ \delta'_c &= 74^\circ 09' 55.108'' \\ \alpha' - \alpha'_c &= \Delta\alpha' = -6^\circ 30' 46.455'' \\ &= -0.113671622 \\ \delta - \delta'_c &= \Delta\delta' = -1^\circ 31' 30.898'' \\ &= -0.26620623\end{aligned}$$

Coefficients for Observation One:

$$\begin{aligned}a_1 &= -0.823772754 \\ a_2 &= -1.254877447 \\ a_3 &= 0.205541131 + 0.332649558 \Delta x - 2.228629851 \Delta y \\ b_1 &= -0.123663802 \\ b_2 &= 0.828501888 \\ b_3 &= -0.666175513 \\ b_4 &= -0.092002565 + 0.745795020 \Delta z \\ b_5 &= 0.155224808 + 0.251217183 \Delta x - 1.683052847 \Delta y\end{aligned}$$

Coefficients for Observation Two:

$$\begin{aligned}a_1 &= -0.962895280 \\ a_2 &= 1.031157127 \\ a_3 &= 0.055127990 - 0.072869565 \Delta x - 1.989118050 \Delta y \\ b_1 &= 0.117108086 \\ b_2 &= 3.196695254 \\ b_3 &= -0.708800652 \\ b_4 &= 0.110352609 - 0.705408840 \Delta z \\ b_5 &= 0.311099780 - 0.411219518 \Delta x - 11.225045283 \Delta y\end{aligned}$$

Equations:

$$\begin{aligned}-0.837361828 \text{ dx} - 1.163835657 \text{ dy} &= 0.008396564 \\-0.120550559 \text{ dx} + 0.807644309 \text{ dy} - 0.696641996 \text{ dz} &= -0.005682042 \\-0.971178482 \text{ dx} + 0.805050853 \text{ dy} &= -0.006266490 \\0.106161166 \text{ dx} + 2.897877564 \text{ dy} - 0.788985620 \text{ dz} &= -0.020825633\end{aligned}$$

A solution of these equations produces the corrections to the approximate coordinates. The units are earth radii which can be converted to meters by multiplying by 6,378,206.4.

$$\begin{aligned}\Delta x &= 0.000297675 \\ \Delta y &= -0.007365136 \\ \Delta z &= -0.000536223\end{aligned}$$

Adding these algebraically we get:

$$\begin{aligned}x_s + \Delta x &= -0.005452325 \\y_s + \Delta y &= 0.782534863 \\z_s + \Delta z &= 0.619413777\end{aligned}$$

The corresponding geodetic coordinates for this first approximation are:

$$\begin{aligned}\phi &= 38^\circ 33' 07.695 \\ \lambda &= 90^\circ 23' 57.130\end{aligned}$$

These values were then used for x_s , y_s and z_s in a second approximation. The resulting corrections and coordinates are:

$$\begin{aligned}\Delta x &= -0.000379847 \\ \Delta y &= 0.000402620 \\ \Delta z &= 0.000553518 \\ x_s + \Delta x &= -0.005832172 \\ y_s + \Delta y &= 0.782937484 \\ z_s + \Delta z &= 0.619967295 \\ \phi &= 38^\circ 33' 45.433 \\ \lambda &= 90^\circ 25' 36.457\end{aligned}$$

A third approximation results in the following values:

$$\Delta x = 0.000000064$$

$$\Delta y = -0.000001631$$

$$\Delta z = 0.000001543$$

$$x_a + \Delta x = -0.005832108$$

$$y_a + \Delta y = 0.782935853$$

$$z_a + \Delta z = 0.619968838$$

$$\phi = 38^\circ 33' 45.89'$$

$$\lambda = 90^\circ 25' 36.44'$$

CONCLUSION

The preceding materials have presented the mathematical procedures for the reduction of artificial satellite observations. The observation of these satellites offers a new opportunity to obtain valuable geodetic information not previously available, and will permit the unification of various continental datums in a single World Geodetic System.

Observations of a satellite can help the geodesist to determine some very basic and essential data. A study of the irregularity of the orbit can reveal data concerning the earth's gravitational field. From investigations of a satellite's period and acceleration, more precise data can be collected from which to determine the size of the earth. The rate of change of the plane of the orbit will help fix the flattening of the earth. Finally, the satellite provides the geodesist with a direct method for connection of datums.

The successful use of earth satellites for geodetic purposes requires a satellite of high stability. It also requires sufficient observation stations to precisely monitor, record and predict the satellite's orbit. Auxiliary observation sites can then be used to gather the desired geodetic data. The accuracies obtainable, as shown in this paper, will permit the use of earth satellite observations for the refinement and extension of the present World Geodetic System.

APPENDIX A

The Ellipse and Spherical Right Triangle

1. Ellipse

a. Axes

$OA = OA' =$ semimajor axis
denoted by the letter a

$OB = OB' =$ semiminor axis
denoted by the letter b

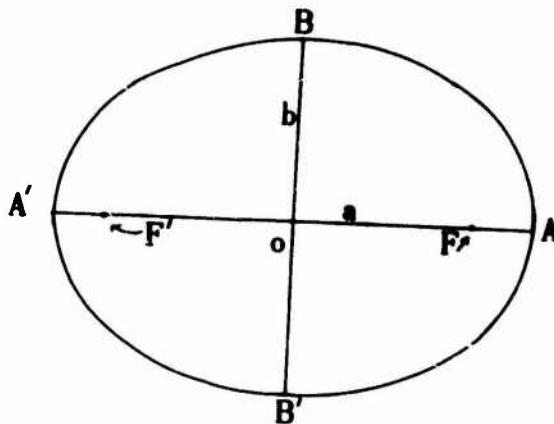


Fig. 1

b. Eccentricity e

$e^2 = \frac{a^2 - b^2}{a^2}$, The value of e determines the divergence of an ellipse from a circle and varies from 0 to 1.

That is: $e = 0$, when $b = a$;

$e = 1$, when $b = 0$.

c. Foci F, F'

d. $\cos v$

$$\cos v = \frac{a(1 - e^2) - R}{eR}$$

Let:

$$KF = m, FP = R$$

Then,

$$MP = m + R \cos \theta.$$

Now, as given by Osgood and Graustein³:

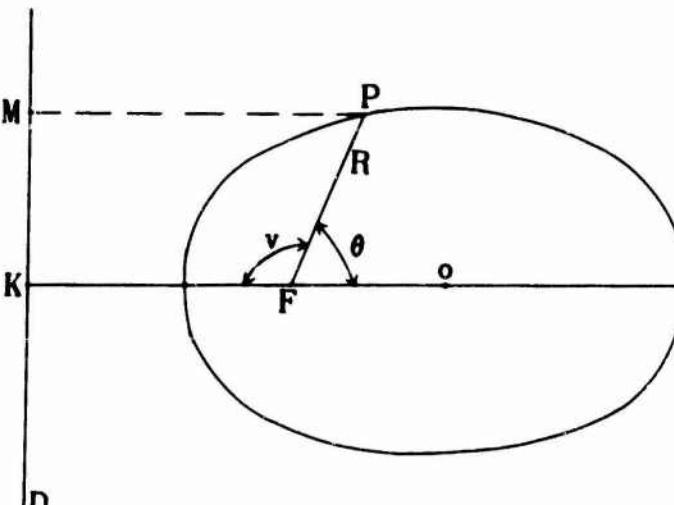


Fig. 2

$$e = FP/MP$$

so,

$$e = \frac{R}{m + R \cos \theta}$$

or,

$$R = \frac{em}{1 - e \cos \theta}$$

Now,

$m = KO - FO$, where by Osgood and Graustein:³

$$KO = a/e, \text{ and } FO = C = ae$$

Then

$$m = \frac{a}{e} - ae = \frac{a}{e}(1 - e^2)$$

and

$$R = \frac{a(1 - e^2)}{1 - e \cos \theta}, \text{ but } \theta = 180 - v, \text{ or } \cos \theta = -\cos v$$

Therefore

$$R = \frac{a(1 - e^2)}{1 + e \cos v}$$

or

$$\cos v = \frac{a(1 - e^2) - R}{e R}$$

2. Spherical Right Triangle and Basic Formulae

a. $\sin a = \sin A \sin c$

b. $\tan b = \cos A \tan c$

where:

angle C is a right angle

and $a = \widehat{BC}$, $b = \widehat{AC}$, $c = \widehat{AB}$

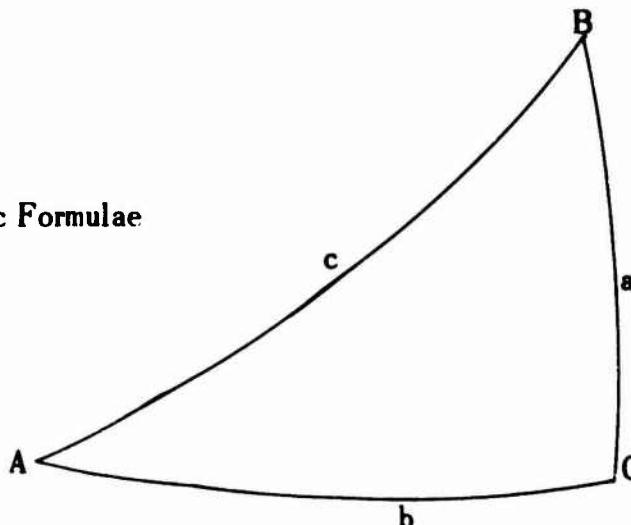


Fig. 3

APPENDIX B

Basic Elements of a Satellite's Orbit

1. Kepler's Laws⁹

- a. The orbit of each planet is an ellipse with the sun at one of its foci.
- b. The radius vector of each planet describes equal areas (areal velocity) in equal times.
- c. The squares of the periods of the planets are proportional to the cubes of their mean distances from the sun.

2. Expression for the areal velocity

Areal velocity is defined as the area swept out by the radius vector (r) in unit time. By Kepler's second law above, it may then be stated that the areal velocity of an earth satellite is constant. Referring to Fig. 4, this means:

triangle SFS' = triangle $S'FS''$ (in area), if the time interval between S and S' and between S' and S'' are equal. F (the foci) is the center of the earth. S , S' , S'' , and S''' are positions of a satellite in its orbit. The r_1 , r_2 , r_3 , and r_4 are radius vectors.

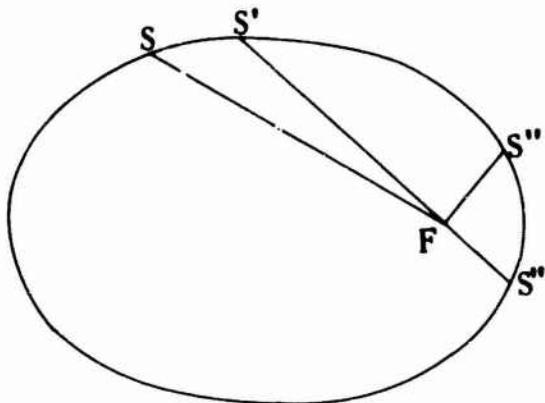


Fig. 4

In Fig. 5:

If SS' is the length of the path described in unit time, then SS' is the linear velocity V .
If the area SFS' is the areal velocity A , then:

$$A = 1/2 V X$$

where,

X = perpendicular from F to the line of motion (FB')

$V = SS'$ or the base of the triangle SFS'

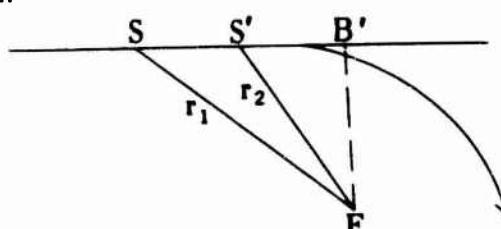


Fig. 5

We next find expressions for V and X .

In Fig. 6:

The earth is at the focus F; $FB = A' O = OA = a$, the semimajor axis. The semiparameter, P, is FY' or FY . The eccentricity, e, is represented by the expression $FO/A' O$. From Russell, Dugan, and Stewart⁹ we have:

$$V^2 = G (M + m) \left(\frac{2}{r} - \frac{1}{a} \right)$$

where,

$$G = \text{constant of gravitation} = 6.668 \times 10^{-8} \text{ dynes}^1$$

$$M = \text{mass of the earth} = 5.983 \times 10^{27} \text{ gm}^1$$

m = mass of the satellite

r = radius vector measured from F to any point on the orbit.

When the satellite is at the extremity of the minor axis, Point B, of the orbit (Fig. 6), $r = FB = a$, and the formula for V becomes:

$$V = \sqrt{\frac{G (M + m)}{a}}$$

From Fig. 6, it is seen that the perpendicular distance X for the satellite at B is equal to FB' , but this is evidently equal to the minor axis b of the ellipse. Hence,

$$A = 1/2 V X = 1/2 b \sqrt{\frac{G (M + m)}{a}}$$

Now, by the geometry of the ellipse: $b = a \sqrt{1 - e^2}$ and P (the semiparameter) =

$$a (1 - e^2) = b \frac{2}{a}$$

The expression for the areal velocity may then be written:

$A = 1/2 P^{\frac{1}{2}} \sqrt{G (M + m)}$, that is, the areal velocity is proportional to the square root of the semiparameter, and since the areal velocity is constant, the above formula applies to any position of the satellite in its orbit.

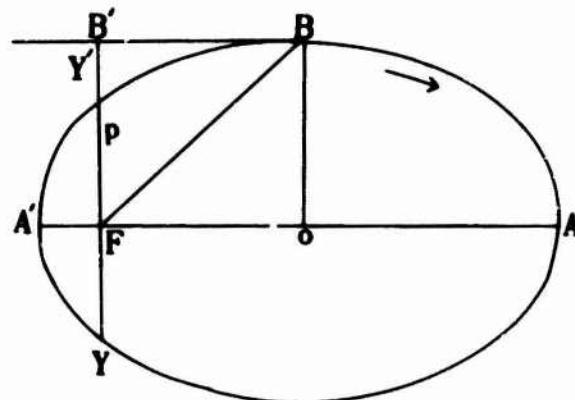


Fig. 6

3. The expression for a satellite's period

The whole area of an ellipse is πab , and since it is swept out by the radius vector at the rate A , the period will be:

$$P = \frac{\pi ab}{A} = \frac{2\pi a^{3/2}}{\sqrt{G(M+m)}}$$

$$\text{let } n = \sqrt{\frac{G(M+m)}{a^3}}$$

$$\text{then } P = \frac{2\pi}{n}$$

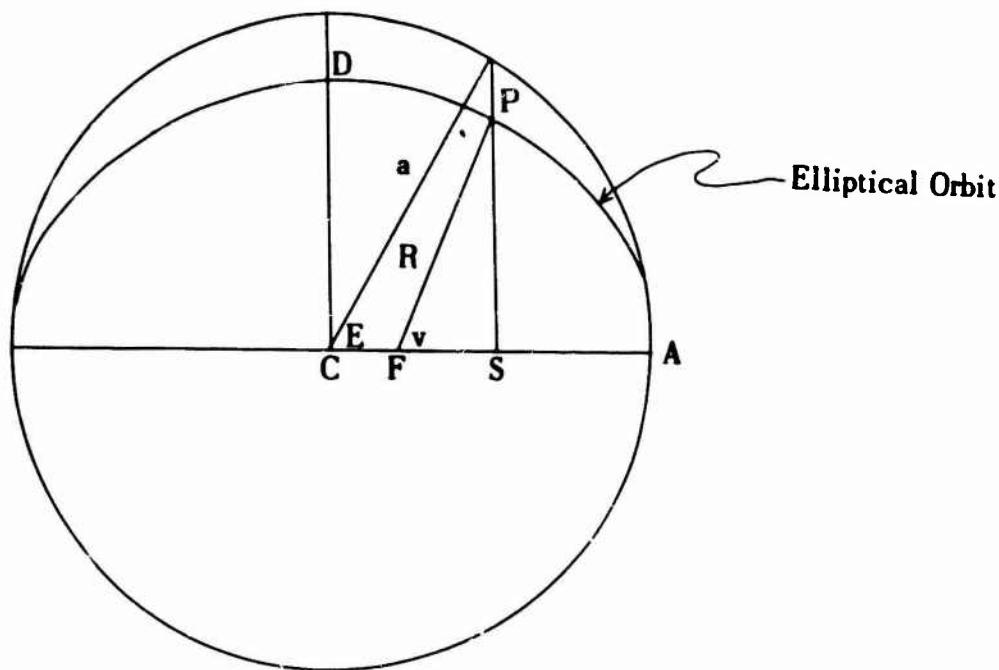


Fig. 7

4. The Eccentric Anomaly (E)⁵

$$PS:QS = b:a$$

$$a = CA = CQ$$

$$b = CD$$

P = Position of satellite in its elliptical orbit

Where:

$$PS = R \sin v, \quad QS = CQ \sin E = a \sin E$$

Then:

$$R = FP \text{ (radius vector)}$$

$$R \sin v : a \sin E = b:a$$

$$v = \text{angle AFP (true anomaly)}$$

$$\text{or} \quad R \sin v = b \sin E$$

(1)

Now:

$$FS = R \cos v, FS = CS - CF = a \cos E - ae$$

Then:

$$R \cos v = a (\cos E - e) \quad (2)$$

$$(1)^2 + (2)^2 = R^2 \sin^2 v + R^2 \cos^2 v = b^2 \sin^2 E + a^2 (\cos E - e)^2$$

$$R^2 = a^2 (1 - e^2) \sin^2 E + a^2 (\cos^2 E - 2 e \cos E + e^2)$$

$$R^2 = a^2 [(1 - e^2) \sin^2 E + \cos^2 E - 2 e \cos E + e^2]$$

$$= a^2 (1 - e^2 \sin^2 E - 2 e \cos E + e^2)$$

$$= a^2 [1 + e^2 (1 - \sin^2 E) - 2 e \cos E]$$

$$= a^2 [1 + e^2 \cos^2 E - 2 e \cos E]$$

$$R = a (1 - e \cos E) \quad (\text{radius vector}) \quad (3)$$

Also:

$$\sin 1/2 v = \pm \sqrt{\frac{1 - \cos v}{2}}$$

$$2 \sin^2 1/2 v = 1 - \cos v$$

$$\begin{aligned} 2 R \sin^2 1/2 v &= R (1 - \cos v) \\ &= a (1 - e \cos E) - a (\cos E - e) \quad \text{see (2)} \\ &= a [1 - e \cos E - \cos E + e] \\ &= a [(1 + e) - \cos E (1 + e)] \end{aligned}$$

$$2 R \sin^2 1/2 v = a (1 + e) (1 - \cos E)$$

$$2 R \cos^2 1/2 v = a (1 - e) (1 + \cos E) \quad (\text{similar procedure})$$

by division

$$\tan^2 1/2 v = \frac{1 + e}{1 - e} \cdot \frac{1 - \cos E}{1 + \cos E}$$

$$\tan v/2 = \left[\frac{1 + e}{1 - e} \right]^{\frac{1}{2}} \cdot \tan E/2 \quad (4)$$

APPENDIX C

The Earth's Oblateness From Satellite Observation

The mean rate of rotation of the orbital plane, or the mean rate of change in the right ascension of the ascending node, Ω , for a near satellite in the gravitational field is represented by the following theoretical expression, according to D. G. King-Hele:⁸

$$n B^2 \cos i [J + J^2 B^2 (\frac{19}{12} \sin^2 i - 1) - \frac{3}{14} D B^2 (7 \sin^2 i - 4)]$$

where:

$$n = \sqrt{\frac{GM}{a^3}}, \text{ the mass } m \text{ of the satellite is insignificant}$$

$$B = \frac{r_e}{a(1 - e^2)}$$

$$J = f - \frac{m}{2} - f \left(\frac{f}{2} - \frac{m}{7} \right)$$

$$D = \frac{f}{2} (7f - 5m)$$

f = Oblateness or flattening of a spheroid representing the Earth

m = the ratio of centrifugal to gravitational acceleration at the equator, multiplied by $(1 - f)$. The value of m is accurately known to be $3449.79 \times 10^{-6} \pm 0.06 \times 10^{-6}$.

The theoretical expression given above has been tested against observed values for the change in Ω during the life-time of Sputnik 2. It was found that the observed values were always slightly lower than the theoretical with the ratio between them remaining virtually constant. The value for the flattening used was 1/297.1.

Since the theoretical values involve the orbital inclination also, tests using information from satellites with different inclinations are necessary. Preliminary tests with three U. S. satellites also indicate the observed values of the change in Ω to be lower than the theoretical. This constant discrepancy seems to be "due mainly to the J-term and that J needs to be reduced."⁸

If m is a constant then the J function depends on the flattening f . Then, if J is to be reduced f must be reduced. In the case of Sputnik 2, $f = 1/298.1 \pm 0.1$ would have been an excellent value.

APPENDIX D

Derivation of Formulae for Photographic Plate Reductions

In order to develop the method of calculating the coordinates of an unknown point from information provided by the photographic image of a portion of the celestial sphere, we must investigate the relationship of the photographic plate, the celestial sphere and the plane tangent to the celestial sphere.

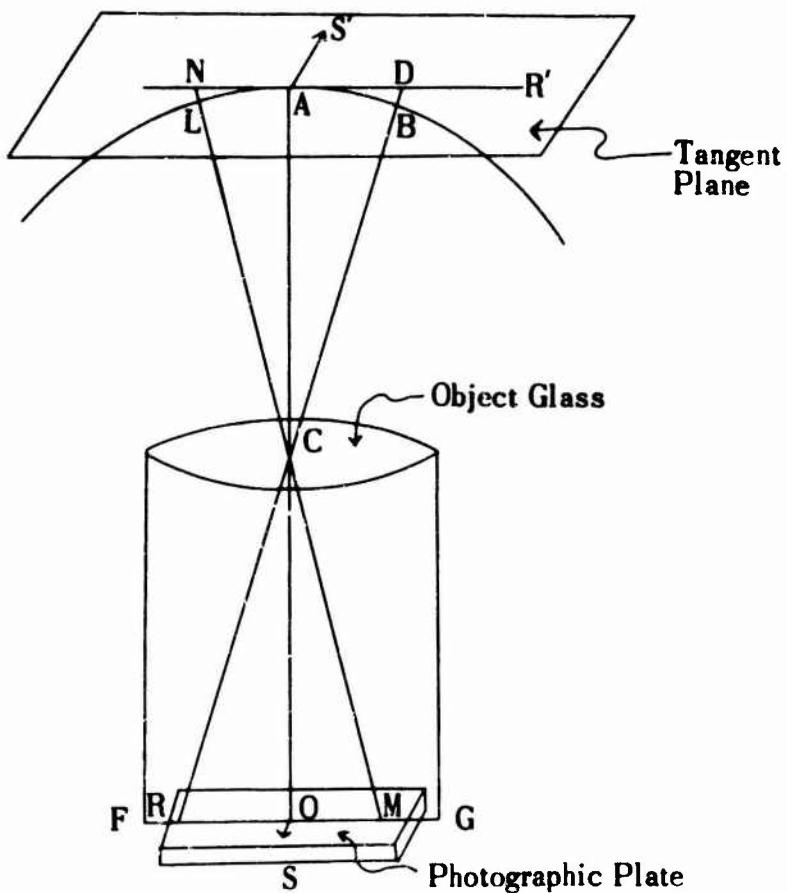


Fig. 8

The optical axis, CO, is perpendicular to the photographic plate. The length CO is the focal length of the camera. The extension of the optical axis intersects the celestial sphere at point A which is also the point where the "Tangent Plane" is tangent to the celestial sphere. The tangent plane is at right angles to CA and is therefore parallel to the photographic plate. Stars at L and B have their projections on the tangent plane at N and D respectively. The image of L appears on the photographic plate at M, and the image of B, at R.

By angular relationships, we can see that the system of images on the photographic plate is similar to the system of projections on the tangent plane. The difference between the two, being the matter of linear scale. If $\phi = OCM = ACL$, then

$$\tan \phi = \frac{OM}{OC} = \frac{AN}{AC} \quad (1)$$

Let AR' , AS' be the positive directions of the rectangular axes in the tangent plane, and let OR , OS be the positive directions of these axes in the plane of the plate. OS is parallel to AS' and OR is parallel to AR' . Note that the positive direction of the axes are opposite to those on the tangent plane.

If we let ξ' and η' be the coordinates of the projection of a star on the tangent plane, and ξ and η the coordinates of the image on the plate; then by similarity we have

$$\frac{\xi'}{AC} = \frac{\xi}{OC} \quad (2)$$

$$\frac{\eta'}{AC} = \frac{\eta}{OC} \quad (3)$$

The terms ξ and η are the coordinates of the image on the plane. In order to determine the relationship between the ξ and η and the astronomical coordinates we shall refer to Fig. 9 where the tangent plane and celestial sphere are tangent at A. (The camera points towards A.)

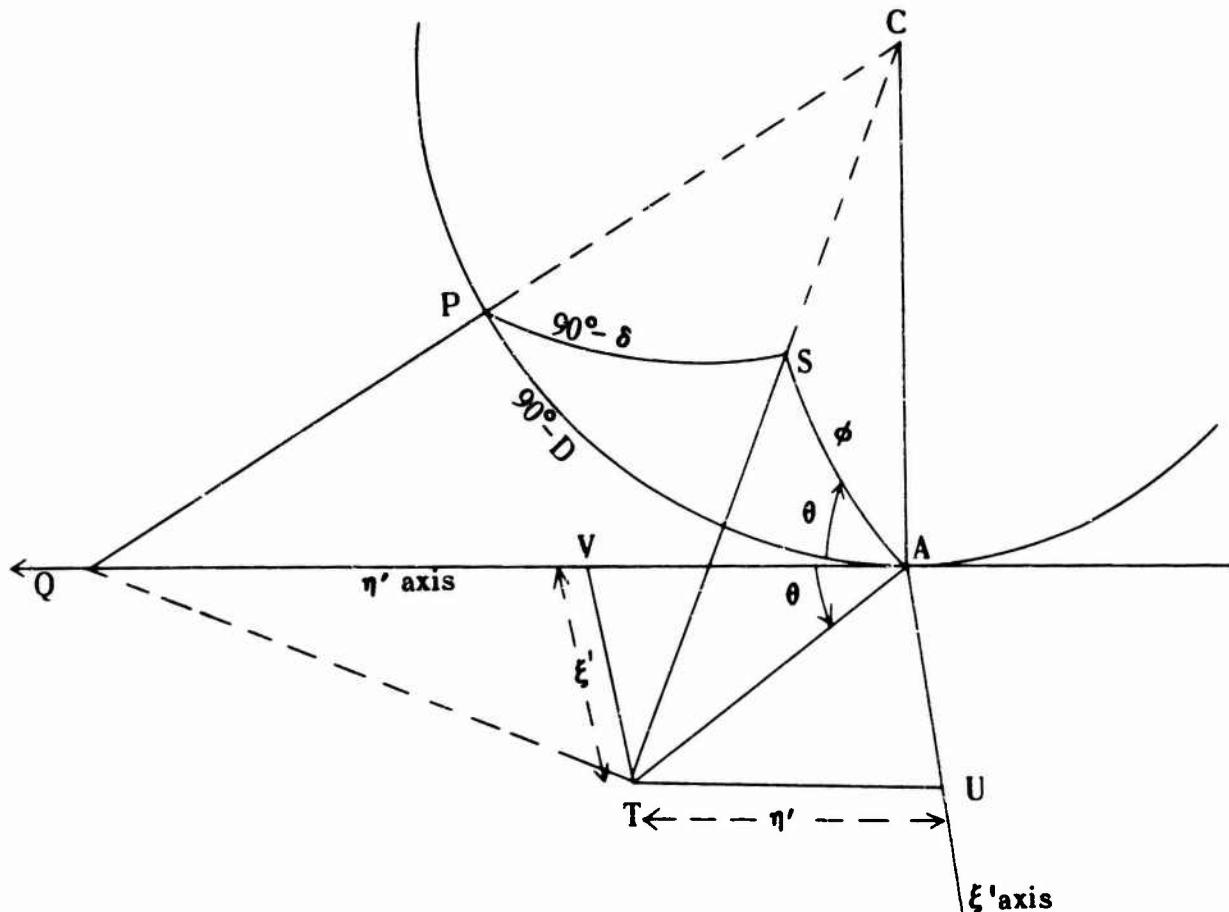


Fig. 9

If S is a star near A , its projection on the tangent plane is obtained by extending the line CS to coincide with the tangent plane at T . We draw the great circle arc AS which projects into a straight line AT on the tangent plane. Let P be the north pole of the celestial sphere. Then AP projects into AQ which we shall call the η' axis. We draw AU perpendicular to AQ , and let this be the ξ' axis. The positive direction of AU is taken to be eastward from the meridian AP , so that increasing values of ξ' correspond to increasing values of right ascension.

Since the arc AP projects into line AQ and arc AS into line AT , and since the sphere is tangent to the plane at A with $AT \perp$ to CA , then QAT defines the spherical angle PAS .

Let arc AS be denoted by ϕ and SAP by θ ; then $QAT = \theta$. Draw perpendiculars TU and TV to AU and AQ respectively. Then

$$VT = \xi' = AT \sin \theta \quad (4)$$

$$UT = \eta' = AT \cos \theta \quad (5)$$

and

$$AT = AC \tan ACT = AC \tan \theta \quad (6)$$

$$\frac{\xi'}{AC} = \tan \phi \sin \theta \quad (7)$$

$$\frac{\eta'}{AC} = \tan \phi \cos \theta \quad (8)$$

$$\frac{\xi}{OC} = \tan \phi \sin \theta \quad (9)$$

$$\frac{\eta}{OC} = \tan \phi \cos \theta \quad (10)$$

In these last two equations, ξ and η are the coordinates of the image of S on the photographic plate with reference to rectangular axes thru O , the center of the plate, and drawn parallel but oppositely directed to the axes AU and AQ on the tangent plane. OC is the focal length. When ξ and η are derived and in the same units as the focal length, then ϕ and θ can be calculated. Since ϕ and θ are functions of the right ascension and declination of A and S , the right ascension and declination of S can be found if the right ascension and declination of A are known.

If we assume the unit of length to be the focal length and ξ and η to be expressed in this unit, then

$$\xi = \tan \phi \sin \theta \quad (11)$$

$$\eta = \tan \phi \cos \theta \quad (12)$$

ξ and η are called the standard coordinates of the star concerned. The use of these formulae assumes that there are no errors involved in the observations. Since a perfect solution is only approached, never reached, a discussion of the errors involved and their effect on the formulae will be included at the end of the discussion of the derivation of the formulae.

Let A_o and D_o be the right ascension and declination of the point A on the sphere, and α and δ the coordinates of star S. The relationship between ξ and η , A_o and D_o , and α and δ can be obtained from the spherical triangle ASP in Fig. 9. Arc $AP = 90^\circ - D$, arc $SP = 90^\circ - \delta$, $ASP = \alpha - A$, arc $AS = \phi$, $SAP = \theta$. From the formulae of spherical trigonometry, we have

$$\cos \phi = \sin \delta \sin D_o + \cos \delta \cos D_o \cos (\alpha - A_o) \quad (13)$$

$$\sin \phi \sin \theta = \cos \delta \sin (\alpha - A_o) \quad (14)$$

$$\sin \phi \cos \theta = \sin \delta \cos D_o - \cos \delta \sin D_o \cos (\alpha - A_o) \quad (15)$$

Since $\eta = \tan \phi \cos \theta$ we can divide (15) by (13) to get

$$\eta = \frac{\cos D_o - \cot \delta \sin D_o \cos (\alpha - A_o)}{\sin D_o + \cot \delta \cos D_o \cos (\alpha - A_o)} \quad (16)$$

For simplification of the computations let

$$\cot q = \cot \delta \cos (\alpha - A_o) \quad (17)$$

Then

$$\eta = \frac{\cos D_o - \sin D_o \cot q}{\sin D_o + \cos D_o \cot q}$$

$$\text{or} \quad \eta = \tan (q - D_o) \quad (18)$$

Since $\xi = \tan \phi \sin \theta$ we can divide 14 by 13 to get

$$\xi = \frac{\cot \delta \sin (\alpha - A_o)}{\sin D_o + \cos D_o \cot \delta \cos (\alpha - A_o)} \quad (19)$$

using q

$$\xi = \frac{\cot q \tan (\alpha - A_o)}{\sin D_o + \cos D_o \cot q} -$$

$$\xi = \frac{\cos q \tan (\alpha - A_o)}{\cos (q - D_o)} \quad (20)$$

derivation of (14)

law of sines

$$\frac{\sin \phi}{\sin (\alpha - A)} = \frac{\cos \delta}{\sin \theta}$$

$$\sin \phi \sin \theta = \cos \delta \sin (\alpha - A)$$

derivation of (13)

law of cosines for sides

$$\cos \phi = \cos (90 - \delta) \cos (90 - D) + \sin (90 - \delta) \sin (90 - D) \cos (\alpha - A)$$

$$\cos \phi = \sin \delta \sin D + \cos \delta \cos D \cos (\alpha - A)$$

derivation of (15)

from law of cosines for sides, find $\cos \theta$

$$\cos (90 - \delta) = \cos \phi \cos (90 - D) + \sin \phi \sin (90 - D) \cos \theta$$

$$\sin \delta - \cos \phi \sin D = \sin \phi \cos D \cos \theta$$

$$\cos \theta = \frac{\sin \delta - \cos \phi \sin D}{\sin \phi \cos D}$$

from (14) above

$$\sin \phi = \frac{\cos \delta \sin (\alpha - A)}{\sin \theta}$$

Multiply $\sin \phi$ and $\cos \theta$

$$\sin \phi \cos \theta = \left[\frac{\cos \delta \sin (\alpha - A)}{\sin \theta} \right] \cdot \left[\frac{\sin \delta - \cos \phi \sin D}{\sin \phi \cos D} \right]$$

$$\sin \phi \cos \theta = \frac{\cos \delta \sin (\alpha - A) [\sin \delta - \cos \phi \sin D]}{\sin \theta \sin \phi \cos D}$$

since $\cos \delta \sin (\alpha - A)$ is equal to $\sin \theta \sin \phi$, by cancelling, we have

$$\sin \phi \cos \theta = \frac{\sin \delta - \cos \phi \sin D}{\cos D}$$

substituting the value of $\cos \phi$

$$\sin \phi \cos \theta = \frac{\sin \delta - \sin D [\sin \delta \sin D + \cos \delta \cos D \cos (\alpha - A)]}{\cos D}$$

$$= \frac{\sin \delta - \sin \delta \sin^2 D - \sin D \cos \delta \cos D \cos (\alpha - A)}{\cos D}$$

$$= \frac{\sin \delta [1 - \sin^2 D] - \sin D \cos \delta \cos D \cos (\alpha - A)}{\cos D}$$

$$\sin \phi \cos \theta = \sin \delta \cos D - \sin D \cos \delta \cos (\alpha - A)$$

To get the formulae in terms of α , δ , A_o , D_o , ξ and η from (16)

$$\begin{aligned}\eta &= \frac{\cos D_o - \cot \delta \sin D_o \cos (\alpha - A_o)}{\sin D_o + \cot \delta \cos D_o \cos (\alpha - A_o)} \\ \eta \sin D_o + \eta \cot \delta \cos D_o \cos (\alpha - A_o) &= \cos D_o - \cot \delta \sin D_o \cos (\alpha - A_o) \\ \cot \delta \cos (\alpha - A_o) [\eta \cos D_o + \sin D_o] &= \cos D_o - \eta \sin D_o \\ \cot \delta \cos (\alpha - A_o) &= \frac{\cos D_o - \eta \sin D_o}{\eta \cos D_o + \sin D_o} \\ \cot \delta \cos (\alpha - A_o) &= \frac{1 - \eta \tan D_o}{\eta + \tan D_o} \quad (21)\end{aligned}$$

from (10)

$$\begin{aligned}\xi &= \frac{\cot \delta \sin (\alpha - A_o)}{\sin D_o + \cos D_o \cot \delta \cos (\alpha - A_o)} \\ \xi [\sin D_o + \cos D_o \cot \delta \cos (\alpha - A_o)] &= \cot \delta \sin (\alpha - A_o) \\ \cot \delta \sin (\alpha - A_o) &= \xi [\sin D_o + \frac{\cos D_o (1 - \eta \tan D_o)}{\eta + \tan D_o}] \\ &= \xi [\sin D_o + \cos D_o - \frac{\eta \cos D_o \frac{\sin D_o}{\cos D_o}}{\eta + \tan D_o}] \\ &= \xi [\sin D_o + \frac{\cos D_o - \sin D_o}{\eta + \tan D_o}] \\ &= \xi [\frac{\eta \sin D_o + \sin D_o \tan D_o + \cos D_o - \eta \sin D_o}{\eta + \tan D_o}] \\ &= \xi [\frac{\sin^2 D_o \sec D_o + \cos D_o}{\eta + \tan D_o}] \\ &= \xi [\frac{\sin D_o \tan D_o + \cos D_o}{\eta + \tan D_o}] \\ \cot \delta \sin (\alpha - A_o) &= \frac{\xi \sec D_o}{\eta + \tan D_o} \quad (22)\end{aligned}$$

Divide (22) by (21)

$$\frac{\cot \delta \sin (\alpha - A_o)}{\cot \delta \cos (\alpha - A_o)} = \frac{\frac{\xi \sec D_o}{\eta + \tan D_o}}{\frac{1 - \eta \tan D_o}{\eta + \tan D_o}}$$
$$\tan (\alpha - A_o) = \frac{\xi \sec D_o}{1 - \eta \tan D_o} \quad (23)$$

APPENDIX E

Discussion of Errors Pertaining to the Photographic Method

The standard coordinates differ from the plate coordinates by small amounts which represent measurable or calculable amounts of imperfection due to a number of errors involved. These errors fall into two categories:

1. Geometrical and Mechanical

- (a) Error of orientation
- (b) Non-perpendicularity of axes
- (c) Centering error
- (d) Error of tilt

2. Spherical

- (a) Refraction
- (b) Aberration

Taken separately, these errors produce in each coordinate a displacement of the image of a star on the plate from the position corresponding to the standard coordinates of the star. This displacement can be represented as a linear expression since a study of these errors reveals that, generally second order effects are negligible.

The linear expression obtained from the combination of all the effects on the coordinates is:

$$\xi - x = ax + by + c \quad (24)$$

$$\eta - y = dx + ey + f \quad (25)$$

The terms a , b , ..., f are called plate constants. The values reflect the errors present in any one plate.

Since the x and y coordinates differ from the ξ and η coordinates by small amounts, the plate constants can be computed from the plate coordinates or from the standard coordinates. The plate coordinates, coming from observational data, have more avenues for errors than the computed standard coordinates; therefore, it is recommended that the plate constants be computed from the form:

$$\xi - x = a\xi + b\eta + c \quad (26)$$

$$\eta - y = d\xi + e\eta + f \quad (27)$$

APPENDIX F

Development of the Method of Dependences

In the working procedure, the method of solving for the plate constants by the least squares is explained. The process requires quite a lengthy computation, and if we have several plates, the plate constants have to be determined from each plate which requires a large amount of numerical work. Consider a series of plates with the same plate-center and in which the position of the unknown point differs little from one plate to the next. The method of dependences was developed mainly for use with such a series of plates. It works, of course, with one plate but nothing is gained in the way of shortening the computation unless it is applied to several plates.

The method of dependences eliminates the specific values of the plate constants and replaces them with terms which express their functions.

The solution will be the same as given in the working instructions up to the point of solution of the normal equations which are:

$$a [\xi_i \xi_i] + b [\xi_i \eta_i] + c [\xi_i] = [\xi_i (\xi_i - x_i)] \quad (1)$$

$$a [\xi_i \eta_i] + b [\eta_i \eta_i] + c [\eta_i] = [\eta_i (\xi_i - x_i)] \quad (2)$$

$$a [\xi_i] + b [\eta_i] + nc = [(\xi_i - x_i)] \quad (3)$$

For the unknown point we have $\xi - x = a\xi + b\eta + c$ (4)

If (ξ_t, η_t) are the standard coordinates of the unknown point for one of the plates (which we shall refer to as the "selected plate"), then we can write (4) as follows:

$$\xi - x = a\xi_t + b\eta_t + c + a(\xi - \xi_t) + b(\eta - \eta_t) \quad (5)$$

It has already been pointed out that the quantities $(\xi - \xi_t)$ and $(\eta - \eta_t)$ are very small quantities and the values of a and b , are also small. Therefore, we can neglect the last two terms, and the equation for the unknown point becomes:

$$\xi - x = a\xi_t + b\eta_t + c \quad (6)$$

We can manage to eliminate a , b , and c from the four equations (1), (2), (3) and (6). Multiply these in order by P , Q , R , and -1 .

Then if P , Q and R are given by:

$$P [\xi_i \xi_i] + Q [\xi_i \eta_i] + R [\xi_i] = \xi_t \quad (7)$$

$$P [\xi_i \eta_i] + Q [\eta_i \eta_i] + R [\eta_i] = \eta_t \quad (8)$$

$$P [\xi_i] + Q [\eta_i] + Rn = 1 \quad (9)$$

we have:

$$\xi - x = P[\xi_i(\xi_i - x_i)] + Q[\eta_i(\xi_i - x_i)] + R[\xi_i - x_i]$$

which can be written

$$\xi - x = \sum_{i=1}^n (P\xi_i + Q\eta_i + R)(\xi_i - x_i) \quad (10)$$

$$\text{let } D_i = P\xi_i + Q\eta_i + R \quad (11)$$

then

$$\xi - x = \sum_{i=1}^n D_i (\xi_i - x_i) \quad (12)$$

Since x differs from ξ by a small amount we can substitute the measured coordinates for the standard coordinates. Thus P , Q and R will be determined from (13), (14) and (15).

$$P[x_i x_i] + Q[x_i y_i] + R[x_i] = x_i \quad (13)$$

$$P[x_i y_i] + Q[y_i y_i] + R[y_i] = y_i \quad (14)$$

$$P[x_i] + Q[y_i] + Rn = 1 \quad (15)$$

Then

$$D_i = Px_i + Qy_i + R \quad (16)$$

The dependences can be formed from (16) and checked by:

$$\sum_{i=1}^n D_i = 1 \quad (17)$$

Then the solution for the standard coordinate ξ can be found from (12).

The procedure is the same for η .

APPENDIX G

Development of Formulae for the Geocentric Coordinates of the Observation Station

In order to derive the formulae which define the relationship between an observer's position and the position of a small earth satellite, we must relate each to an earth-centered set of axes.

Let the axes X, Y, Z be defined so that the Z axis points toward the north pole, the X axis points toward the vernal equinox and the Y axis is six hours eastward from the vernal equinox.¹⁵

Another set of axes, x, y, z are centered at the same point, and the z axis coincides with the Z axis of the first system. The x axis points toward the Greenwich meridian, and the y axis is located 90° westward from the Greenwich meridian.¹⁴

Consider the observer's position, K, at a distance, ρ , from the center or origin, C. In order to obtain the geodetic coordinates x_o , y_o , z_o , we get the relationships from the geometry. Let K' be the projection of K in the xy plane. Then the distance that CK' is from the x axis is the angle λ' . Then let ϕ' be the angle between CK' and CK. In other words, ϕ' , λ' , and ρ are the geocentric coordinates of K in polar form.

$$\text{Then } z_o = \rho \sin \phi'$$

$$y_o = \rho \cos \phi' \sin \lambda'$$

$$x_o = \rho \cos \phi' \cos \lambda' \quad (1)$$

If X_1 , Y_1 , Z_1 are the coordinates of K in the celestial system, we find

$$Z_1 = \rho \sin \phi'$$

$$Y_1 = \rho \cos \phi' \sin \theta$$

$$X_1 = \rho \cos \phi' \cos \theta \quad (2)$$

where θ is the hour angle of the vernal equinox with respect to the geocentric meridian at K.

The hour angle of the vernal equinox with respect to the geocentric meridian at Greenwich is γ .

$$\text{Then } \gamma = \theta + \lambda'$$

Substituting in (2) we get

$$\begin{aligned}
 X_1 &= \rho \cos \phi' \cos (\gamma - \lambda') \\
 X_1 &= \rho \cos \phi' (\cos \gamma \cos \lambda' + \sin \gamma \sin \lambda') \\
 X_1 &= x_o \cos \gamma + y_o \sin \gamma \\
 Y_1 &= \rho \cos \phi' \sin (\gamma - \lambda') \\
 Y_1 &= \rho \cos \phi' (\sin \gamma \cos \lambda' - \cos \gamma \sin \lambda') \\
 Y_1 &= x_o \sin \gamma - y_o \cos \gamma \\
 Z_1 &= z_o
 \end{aligned} \tag{3}$$

If α and δ are the geocentric right ascension and declination of the satellite, and R the distance from the center of the earth to the center of the satellite, then the celestial geocentric coordinates of the satellite are

$$\begin{aligned}
 X_2 &= R \cos \delta \cos \alpha \\
 Y_2 &= R \cos \delta \sin \alpha \\
 Z_2 &= R \sin \delta
 \end{aligned} \tag{4}$$

If r is the distance from K to S and α' , δ' are the apparent coordinates of S as seen from K , we have from geometric relationships

$$\begin{aligned}
 X_2 - X_1 &= r \cos \delta' \cos \alpha' \\
 Y_2 - Y_1 &= r \cos \delta' \sin \alpha' \\
 Z_2 - Z_1 &= r \sin \delta'
 \end{aligned} \tag{5}$$

These equations are the basic ones which relate the apparent coordinates of the satellite, the geocentric coordinates of the satellite, and the geocentric coordinates of the observer's position.

We must put equations (5) into a form which uses the observed data and computed data and can be solved for the observer's coordinates.

If we multiply the sides of the first equation of (5) by $(+\sin \alpha')$ we have

$$\sin \alpha' (X_2 - X_1) = \sin \alpha' r \cos \delta' \cos \alpha'$$

Now if we substitute the values for X_2 and X_1 from (4) and (3) respectively we get

$$\sin \alpha' [(R \cos \delta \cos \alpha) - (x_o \cos \gamma + y_o \sin \gamma)] = \sin \alpha' r \cos \delta' \cos \alpha'$$

Next we take the second equation of (5) and multiply both sides by $(-\cos \alpha')$

$$-\cos \alpha' (Y_2 - Y_1) = -\cos \alpha' r \cos \delta' \sin \alpha'$$

and substituting from (4) and (3)

$$-\cos \alpha' [(R \cos \delta \sin \alpha) - (x_o \sin \gamma - y_o \cos \gamma)] = -\cos \alpha' r \cos \delta' \sin \alpha'$$

We multiply both sides of the third equation of (5) by $\sin \alpha'$.

Combining these two equations algebraically we have

$$\begin{aligned} \sin \alpha' [(R \cos \delta \cos \alpha) - (x_o \cos \gamma + y_o \sin \gamma)] - \cos \alpha' [(R \cos \delta \sin \alpha) \\ - (x_o \sin \gamma - y_o \cos \gamma)] &= \sin \alpha' r \cos \delta' \cos \alpha' - \cos \alpha' r \cos \delta' \sin \alpha' \\ \sin \alpha' R \cos \delta \cos \alpha - \sin \alpha' x_o \cos \gamma - \sin \alpha' y_o \sin \gamma - \cos \alpha' R \cos \delta \sin \alpha \\ + \cos \alpha' x_o \sin \gamma - \cos \alpha' y_o \cos \gamma &= \sin \alpha' r \cos \delta \cos \alpha' - \cos \alpha' r \cos \delta' \sin \alpha' \end{aligned}$$

Combining terms we get

$$x_o \sin(\alpha' - \gamma) + y_o \cos(\alpha' - \gamma) = R \cos \delta \sin(\alpha' - \alpha) \quad (6)$$

Again using the equations in (5), multiply the first equation by $(+\cos \alpha')$, the second by $(+\sin \alpha')$, and the third by $(-\cos \delta')$

$$\cos \alpha' r \cos \delta' \cos \alpha' = \cos \alpha' R \cos \delta \cos \alpha - \cos \alpha' x_o \cos \gamma - \cos \alpha' y_o \sin \gamma$$

$$\begin{aligned} \sin \alpha' r \cos \delta' \sin \alpha' &= \sin \alpha' R \cos \delta \sin \alpha - \sin \alpha' x_o \sin \gamma + \sin \alpha' y_o \cos \gamma \\ &\quad - \cot \delta' r \sin \delta' = -\cot \delta' R \sin \delta + \cot \delta' z_o \end{aligned}$$

$$r \cos \delta' (\sin^2 \alpha' + \cos^2 \alpha') - r \sin \delta' \cot \delta' = R \cos \delta (\sin \alpha' \sin \alpha + \cos \alpha' \cos \alpha)$$

$$- R \sin \delta \cot \delta' - x_o (\sin \alpha' \sin \gamma + \cos \alpha' \cos \gamma) + y_o (\sin \alpha' \cos \gamma - \cos \alpha' \sin \gamma) + z_o \cot \delta'$$

$$\begin{aligned} r \cos \delta' - r \sin \delta' \cos \delta' &= R \cos \delta \cos(\alpha' - \alpha) - R \sin \delta \cos \delta' - x_o \cos(\alpha' - \gamma) \\ &\quad + y_o \sin(\alpha' - \gamma) + z_o \cot \delta' \end{aligned}$$

$$R \sin \delta \cot \delta' - R \cos \delta \cos(\alpha' - \alpha) = x_o \cos(\alpha' - \gamma) + y_o \sin(\alpha' - \gamma) + z_o \cos \delta'$$

Or

$$x_o \cos(\alpha' - \gamma) - y_o \sin(\alpha' - \gamma) - z_o \cos \delta' = R \cos \delta \cos(\alpha' - \alpha) - R \sin \delta \cot \delta' \quad (7)$$

If we let $G' = \alpha' - \gamma$, then equations (6) and (7) become

$$x_o \sin G' + y_o \cos G' = R \cos \delta \sin(\alpha' - \alpha)$$

$$x_o \cos G' - y_o \sin G' - z_o \cot \delta' = R \cos \delta \cos(\alpha' - \alpha) - R \sin \delta \cot \delta'$$

These equations will produce the x, y, z coordinates of a station which have before been unknown. In the case of a station that is located by approximate coordinates, it is possible to provide a solution which gives corrections to the position. The development of the formulae begins again with the equations from (5). First we multiply each respectively by (- sin α), (+ cos α) and (0).

$$-r \cos \delta' \cos \alpha' \sin \alpha = -R \cos \delta \cos \alpha \sin \alpha + x_0 \cos \gamma \sin \alpha + y_0 \sin \gamma \sin \alpha$$

$$r \cos \delta' \sin \alpha' \cos \alpha = R \cos \delta \sin \alpha \cos \alpha - x_0 \sin \gamma \cos \alpha + y_0 \cos \gamma \cos \alpha$$

Combining these equations, we have

$$r \cos \delta' \sin \alpha' \cos \alpha - r \cos \delta' \cos \alpha' \sin \alpha = R \cos \delta \sin \alpha \cos \alpha$$

$$-R \cos \delta \cos \alpha \sin \alpha + x_0 \cos \gamma \sin \alpha - x_0 \sin \gamma \cos \alpha + y_0 \sin \gamma \sin \alpha + y_0 \cos \gamma \cos \alpha$$

$$r \cos \delta' (\sin \alpha' \cos \alpha - \cos \alpha' \sin \alpha) = x_0 (\cos \gamma \sin \alpha - \sin \gamma \cos \alpha)$$

$$+ y_0 (\sin \gamma \sin \alpha + \cos \gamma \cos \alpha)$$

$$r \cos \delta' \sin (\alpha' - \alpha) = x_0 \sin (\alpha - \gamma) + y_0 \cos (\alpha - \gamma) \quad (8)$$

Next we multiply each equation from (5) by (+ cos α), (+ sin α) and (0)

$$r \cos \delta' \cos \alpha' \cos \alpha = R \cos \delta \cos^2 \alpha - x_0 \cos \gamma \cos \alpha - y_0 \sin \gamma \cos \alpha$$

$$r \cos \delta' \sin \alpha' \sin \alpha = R \cos \delta \sin^2 \alpha - x_0 \sin \gamma \sin \alpha + y_0 \cos \gamma \sin \alpha$$

Combining we get

$$r \cos \delta' (\cos \alpha' \cos \alpha + \sin \alpha' \sin \alpha) = R \cos \delta (\cos^2 \alpha + \sin^2 \alpha)$$

$$- x_0 (\cos \gamma \cos \alpha + \sin \gamma \sin \alpha) + y_0 (\cos \gamma \sin \alpha - \sin \gamma \cos \alpha)$$

Or

$$r \cos \delta' \cos (\alpha' - \alpha) = R \cos \delta - x_0 \cos (\alpha - \gamma) + y_0 \sin (\alpha - \gamma) \quad (9)$$

Dividing (8) by (9) and clearing fractions

$$\frac{r \cos \delta' \sin (\alpha' - \alpha)}{r \cos \delta' \cos (\alpha' - \alpha)} = \frac{x_0 \sin (\alpha - \gamma) + y_0 \cos (\alpha - \gamma)}{x_0 \cos (\alpha - \gamma) + y_0 \sin (\alpha - \gamma) + R \cos \delta}$$

$$\tan (\alpha' - \alpha) = \frac{x_0 \sin (\alpha - \gamma) + y_0 \cos (\alpha - \gamma)}{-x_0 \cos (\alpha - \gamma) + y_0 \sin (\alpha - \gamma) + R \cos \delta}$$

$$-x_0 \cos (\alpha - \gamma) \tan (\alpha' - \alpha) + y_0 \sin (\alpha - \gamma) \tan (\alpha' - \alpha) + R \cos \delta \tan (\alpha' - \alpha) \\ = x_0 \sin (\alpha - \gamma) + y_0 \cos (\alpha - \gamma)$$

$$x_0 [\cos (\alpha - \gamma) \tan (\alpha' - \alpha) + \sin (\alpha - \gamma)]$$

$$+ y_0 [-\sin (\alpha - \gamma) \tan (\alpha' - \alpha) + \cos (\alpha - \gamma)] = R \cos \delta \tan (\alpha' - \alpha) \quad (10)$$

Dividing the third equation of (5) by (9)

$$\begin{aligned}
 \frac{r \sin \delta'}{r \cos \delta'} &= \frac{\cos(\alpha' - \alpha)(z_2 - z_1)}{R \cos \delta - x_0 \cos(\alpha - \gamma) + y_0 \sin(\alpha - \gamma)} \\
 \tan \delta' &= \frac{\cos(\alpha' - \alpha)(R \sin \delta - z_0)}{-x_0 \cos(\alpha - \gamma) + y_0 \sin(\alpha - \gamma) + R \cos \delta} \\
 -x_0 \cos(\alpha - \gamma) \tan \delta' + y_0 \sin(\alpha - \gamma) \tan \delta' + R \cos \delta \tan \delta' \\
 &= \cos(\alpha' - \alpha)(R \sin \delta - z_0) \\
 x_0 \cos(\alpha - \gamma) \tan \delta' - y_0 \sin(\alpha - \gamma) \tan \delta' - R \cos \delta \tan \delta' \\
 &= -\cos(\alpha' - \alpha)(R \sin \delta - z_0) \quad (11)
 \end{aligned}$$

If we let $G = \alpha - \gamma$, then equations (10) and (11) become:

$$\begin{aligned}
 x_0 [\sin G + \tan(\alpha' - \alpha) \cos G] + y_0 [\cos G - \tan(\alpha' - \alpha) \sin G] \\
 - R \cos \delta \tan(\alpha' - \alpha) = 0 \quad (12)
 \end{aligned}$$

$$\begin{aligned}
 x_0 \cos G \tan \delta' - y_0 \sin G \tan \delta' - z_0 \cos(\alpha' - \alpha) \\
 + R \cos(\alpha' - \alpha) \sin \delta - R \cos \delta \tan \delta' = 0 \quad (13)
 \end{aligned}$$

Let x_s, y_s, z_s be provisionally adopted coordinates of K so that

$$x_0 = x_s + \Delta x, y_0 = y_s + \Delta y, z_0 = z_s + \Delta z$$

Let α'_c, δ'_c be the apparent coordinates obtained through use of the values x_s, y_s, z_s above so that

$$\alpha' = \alpha'_c + \Delta \alpha', \quad \delta' = \delta'_c + \Delta \delta'$$

or

$$\Delta \alpha' = \alpha' - \alpha'_c, \quad \Delta \delta' = \delta' - \delta'_c$$

In the following development, we will assume the differentials $dx, dy, dz, d\alpha'$ and $d\delta'$ to equal $\Delta x, \Delta y, \Delta z, \Delta \alpha'$ and $\Delta \delta'$ respectively, giving:

$$\begin{aligned}
 x_0 = x_s + dx, y_0 = y_s + dy, z_0 = z_s + dz \\
 d\alpha' = \alpha' - \alpha'_c, \quad d\delta' = \delta' - \delta'_c \quad (14)
 \end{aligned}$$

Differentiating equations (12) and (13) with respect to x_0, y_0, z_0, α' and δ' , which may vary, we obtain:

from (12)

$$\frac{df}{dx} = \sin G + \tan(\alpha' - \alpha) \cos G$$

$$\frac{df}{dy} = \cos G - \tan(\alpha' - \alpha) \sin G$$

$$\frac{df}{d\alpha'} = \sec^2(\alpha' - \alpha) [-R \cos \delta + x_o \cos G - y_o \sin G]$$

from (13)

$$\frac{df}{dx} = \tan \delta' \cos G$$

$$\frac{df}{dy} = -\tan \delta' \sin G$$

$$\frac{df}{dz} = -\cos(\alpha' - \alpha)$$

$$\frac{df}{d\alpha'} = \sin(\alpha' - \alpha) [-R \sin \delta + z_o]$$

$$\frac{df}{d\delta'} = \sec^2 \delta' [-R \cos \delta + x_o \cos G - y_o \sin G]$$

Finally we use these derivatives to form two differential equations, namely:

$$a_1 dx + a_2 dy = a_3 d\alpha' \quad (15)$$

$$b_1 dx + b_2 dy + b_3 dz = b_4 d\alpha' + b_5 d\delta' \quad (16)$$

where:

$$a_1 = \sin G + \tan(\alpha' - \alpha) \cos G$$

$$a_2 = \cos G - \tan(\alpha' - \alpha) \sin G$$

$$a_3 = \sec^2(\alpha' - \alpha) [R \cos \delta - (x_o + dx) \cos G + (y_o + dy) \sin G]$$

$$b_1 = \tan \delta' \cos G$$

$$b_2 = -\tan \delta' \sin G$$

$$b_3 = -\cos(\alpha' - \alpha)$$

$$b_4 = \sin(\alpha' - \alpha) [R \sin \delta - (z_o + dz)]$$

$$b_5 = \sec^2 \delta' [R \cos \delta - (x_o + dx) \cos G + (y_o + dy) \sin G]$$

APPENDIX H

Relationship Between Geocentric, Geodetic, and x,y,z Space Coordinates

- I. The x, y, z space coordinate formulae are:

$$x = \rho \cos \phi' \cos \lambda$$

$$y = \rho \cos \phi' \sin \lambda$$

$$z = \rho \sin \phi'$$

where:

$$\rho = a (1 - e^2 \sin^2 \theta)^{1/2}$$

$$\tan \phi' = (1 - e^2) \tan \phi$$

II. Explanation of Notations

1. ϕ, λ - geodetic coordinates (latitude and longitude)
2. ρ - geocentric radius
3. ϕ' - geocentric latitude
4. a - semimajor axis of the reference ellipsoid
5. e - eccentricity of the reference ellipsoid
6. θ - reduced latitude

$$\tan \theta = (1 - f) \tan \phi$$

7. f - flattening of the reference ellipsoid
8. The xz plane contains the Greenwich meridian

The xy plane is the equatorial plane; the positive y-axis being 90° west of the x-axis.

III. Elements of the Clarke (1866) ellipsoid for use with stations on North American Datum

$$a = 6,378,206.4 \text{ meters}$$

$$e^2 = 0.00676866$$

$$f = 0.00339008$$

IV. To find geodetic coordinates from x, y, z space coordinates

$$\tan \lambda = y/x$$

$$\tan \phi = \frac{\tan \phi'}{1-e^2}$$

where:

$$\sin \phi' = z/\rho$$

$$\rho = (x^2 + y^2 + z^2)^{1/2}$$

V. Numerical Example

Given:

$$\phi = 35^\circ \text{ N}$$

$$\lambda = 80^\circ \text{ W}$$

Assume the point to be on North American Datum, then:

$$a = 6,378,206.4 \text{ meters}$$

$$e^2 = 0.00676866$$

$$f = 0.00339008$$

Tabulation:

$$\tan \phi = 0.70020755$$

$$1 - e^2 = 0.99323134$$

$$\tan \phi' = 0.69546808$$

$$\sin \phi' = 0.57096264$$

$$\cos \phi' = 0.82097604$$

$$\sin \lambda = 0.98480775$$

$$\cos \lambda = 0.17364818$$

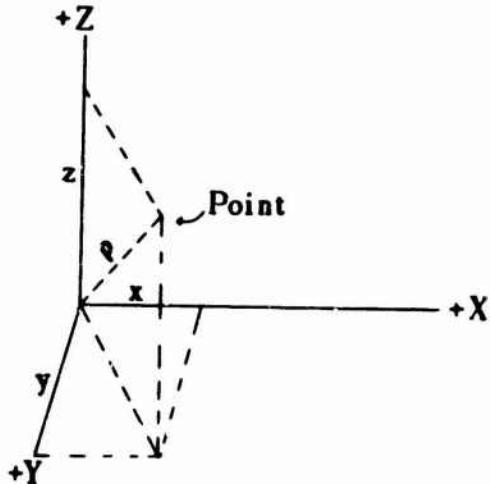


Fig. 10

$$1 - f = 0.9960992$$

$$\tan \theta = 0.39783379$$

$$\sin \theta = 0.32749238$$

$$\sin^2 \theta = 0.32749238$$

$$e^2 \sin^2 \theta = 0.00221668$$

$$(1-e^2 \sin^2 \theta)^{1/2} = 0.998891045$$

$$\rho = 6.37113326 \times 10^6$$

Computation of x, y, z

$$x = 5.23054775 \times 10^6 \times 0.17364818 = 908,275.10 \text{ meters}$$

$$y = 5.23054775 \times 10^6 \times 0.98480775 = 5,151,083.96 \text{ meters}$$

$$z = 6.37113326 \times 10^6 \times 0.57096264 = 3,637,679.07 \text{ meters}$$

VI. Reverse Check

Given:

$$x = 908,275.10 \text{ meters}$$

$$y = 5,151,083.96 \text{ meters}$$

$$z = 3,637,679.07 \text{ meters}$$

Computation of λ , ρ , ϕ' , and ϕ

$$\tan \lambda = y/x = 5.07128171$$

$$\lambda = 80^\circ \text{ W}$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$\sin \phi' = z/\rho = 0.57096264$$

$$\rho^2 = 0.4059133864 \times 10^{14}$$

$$\tan \phi' = 0.69546809$$

$$\rho = 6.37113323 \times 10^6$$

$$1/(1-e^2) = 1.00681479$$

$$\tan \phi = 0.70020756$$

$$\phi = 35^\circ \text{ N}$$

REFERENCES

1. Davis, R. J., Whipple, F. L., and Zirker, J. B., "The Orbit of a Small Earth Satellite," reprinted from Scientific Uses of Earth Satellites, J. S. Van Allen, Univ. of Mich. Press, 1956.
2. Moulton, F. R., An Introduction to Celestial Mechanics, The Macmillan Co., 1931.
3. Osgood, W. F., and Graustein, W. C., Plane and Solid Analytic Geometry, The Macmillan Co., 1921.
4. Smart, W. M., Text-Book on Spherical Astronomy, Cambridge at the University Press, 1936.
5. Ashbrook, J., Schilling, G. F., and Sterne, T. E., "Glossary of Astronomical Terms for the Description of Satellite Orbits," Smithsonian Institution Astrophysical Observatory, 1957.
6. Boss, Benjamin, General Catalogue of 33342 Stars for the Epoch 1950.
7. Sterne, T. E., "Celestial Mechanics of Artificial Satellites," reprinted from Sky and Telescope, Vol. XVII, No. 2, 1957.
8. Merson, R. H., and King-Helle, D. G., "Use of Artificial Satellites to Explore the Earth's Gravitational Field; Results from Sputnik 2 (1957 B)," reprinted from Nature, Vol. 182, 1958.
9. Russell, H. N., Dugan, R. S., and Stewart, J. Q., Astronomy, Ginn and Co., 1945.
10. Chauvenet, W., A Manual of Spherical and Practical Astronomy, J. B. Lippincott Co., 1893.
11. Newcomb, S., A Compendium of Spherical Astronomy, The Macmillan Co., 1906.
12. Hoskinson, A. J., and Duerksen, J. A., Manual of Geodetic Astronomy, Special Pub. No. 237, U. S. Dept. of Commerce, Coast and Geodetic Survey, 1952.
13. Leighton, H. L. C., Solid Geometry and Spherical Trigonometry, D. Van Nostrand Co., Inc., 1943.
14. Markowitz, W., "Geocentric Coordinates From Lunar and Satellite Observations," U. S. Naval Observatory, 1957.
15. Doolittle, C. L., A Treatise on Practical Astronomy, John Wiley and Sons, New York, 1903.